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## Uniform estimates for positive solutions of a class of semilinear elliptic equations and related Liouville and one-dimensional symmetry results

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# UNIFORM ESTIMATES FOR POSITIVE SOLUTIONS OF A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS AND RELATED LIOUVILLE AND ONE-DIMENSIONAL SYMMETRY RESULTS

CHRISTOS SOURDIS

Abstract. We consider the semilinear elliptic equation $\Delta u = W'(u)$ with Dirichlet boundary conditions in a smooth, possibly unbounded, domain $\Omega \subset \mathbb{R}^n$. Under suitable assumptions on the potential $W$, including the double well potential that gives rise to the Allen-Cahn equation, we deduce a condition on the size of the domain that implies the existence of a positive solution satisfying a uniform pointwise estimate. Here, uniform means that the estimate is independent of $\Omega$. The main advantage of our approach is that it allows us to remove a restrictive monotonicity assumption on $W$ that was imposed in the recent paper by G. Fusco, F. Leonetti and C. Pignotti [134]. In addition, we can remove a non-degeneracy condition on the global minimum of $W$ that was assumed in the latter reference. Furthermore, we can generalize an old result of P. Hess [153] and D. G. De Figueiredo [100], concerning semilinear elliptic nonlinear eigenvalue problems. Moreover, we study the boundary layer of global minimizers of the corresponding singular perturbation problem. For the above applications, our approach is based on a refinement of a useful result that dates back to P. Clément and G. Sweers [86], concerning the behavior of global minimizers of the associated energy over large balls, subject to Dirichlet conditions. Combining this refinement with global bifurcation theory and the celebrated sliding method, we can prove uniform estimates for solutions away from their nodal set, refining a lemma from a well known paper of H. Berestycki, L. A. Caffarelli and L. Nirenberg [38]. In particular, combining our approach with a-priori estimates that we obtain by blow-up, the doubling lemma of P. Polacik, P. Quittner, and P. Souplet [206] and known Liouville type theorems, we can give a new proof of a Liouville type theorem of Y. Du and L. Ma [108], without using boundary blow-up solutions. We can also provide an alternative proof, and a useful extension, of a Liouville theorem of H. Berestycki, F. Hamel, and H. Matano [45], involving the presence of an obstacle. Making use of the latter extension, we consider the singular perturbation problem with mixed boundary conditions. Furthermore, we prove some new one-dimensional symmetry properties of certain entire solutions to Allen-Cahn type equations, by exploiting for the first time an old result of Caffarelli, Garofalo, and Segála [74], and we suggest a connection with the theory of minimal surfaces. Using this approach, we can give a new proof of Gibbons' conjecture which is a weak form of the famous conjecture of De Giorgi. Furthermore, we provide new proofs of well known symmetry results in half-spaces with Dirichlet boundary conditions. Moreover, we can generalize a rigidity result due to A. Farina [120]. Lastly, we study the one-dimensional symmetry of solutions in convex cylindrical domains with Neumann boundary conditions.

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## 1. Introduction and statement of the main result

A problem that has received considerable attention in the literature is the study of the structure of solutions $(\lambda, u) \in \mathbb{R} \times C^{2,\alpha}(\bar{\mathcal{D}})$, $0 < \alpha < 1$, depending on the nonlinearity $f$, of the semilinear elliptic nonlinear eigenvalue problem

$$\Delta u + \lambda f(u) = 0,\ x \in \mathcal{D};\quad u(x) = 0,\ x \in \partial\mathcal{D}, \tag{1.1}$$

where $\mathcal{D}$ is typically a smooth bounded domain. To this end, the main approaches used include the method of upper and lower solutions, bifurcation techniques, as well as topological and variational methods (see [169], [184], [228], [232] and the references therein).

Recently, G. Fusco, F. Leonetti and C. Pignotti considered in [134] the semilinear elliptic problem

$$\begin{cases} \Delta u = W'(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a domain with nonempty Lipschitz boundary (see for instance [115]), under the following assumptions on the $C^2$ function $W : \mathbb{R} \to \mathbb{R}$, which we will often refer to as a potential:

**(a):** There exists a constant $\mu > 0$ such that

$$0 = W(\mu) < W(t), \ t \in [0,\infty), \ t \neq \mu,$$
$$W(-t) \geq W(t), \ t \in [0,\infty);$$

**(b):** $W'(t) \leq 0, \ t \in (0,\mu)$;

**(c):** $W''(\mu) > 0$.

A model potential which satisfies the assumptions in [134] is the double well potential in (1.23) below, appearing frequently in the mathematical study of phase transitions, see [102]. Another, model example is given in (4.1). An example of an unbounded domain with nonempty Lipschitz boundary is (4.10) below, which was considered in [38]. We stress that, in the case where the domain is unbounded, the boundary conditions in (1.2) *do not* refer to $u(x) \to 0$ as $|x| \to \infty$ with $x \in \Omega$. Note that (1.1) can be related to (1.2) via a simple rescaling (see the relation between (7.9) and (7.10) below).

***Some preliminary notation.*** For $x \in \mathbb{R}^n$, $\rho > 0$, we let

$$B_\rho(x) = \{y \in \mathbb{R}^n \ : \ |y-x| < \rho\}, \quad B_\rho = B_\rho(0),$$
$$A + B = \{x + y \ : \ x \in A, \ y \in B\}, \quad A, B \subset \mathbb{R}^n,$$

and denote by $d(x,E)$ the Euclidean distance of the point $x \in \mathbb{R}^n$ from the set $E \subset \mathbb{R}^n$, and by $|E|$, unless specified otherwise, the $n$-dimensional Lebesgue measure of $E$ (see [115]). By $\mathcal{O}(\cdot)$, $o(\cdot)$ we will denote the standard Landau's symbols.

The main result of [134] was the following:

**Theorem 1.1.** Assume $\Omega$ and $W$ as above. There are positive constants $R^*$, $r^* \in (0,R^*)$, $a^* \in (0,\mu)$, $k$, $K$, depending only on $W$ and $n$, such that if $\Omega$ contains a closed ball of radius $R^*$, then problem (1.2) has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ verifying

$$0 < u(x) < \mu, \quad x \in \Omega, \tag{1.3}$$

$$\mu - a^* < u(x), \quad x \in \Omega_{R^*} + B_{r^*}, \tag{1.4}$$

and

$$\mu - u(x) \leq K e^{-kd(x,\partial\Omega)}, \quad x \in \Omega, \tag{1.5}$$

where

$$\Omega_{R^*} = \{x \in \Omega \ : \ d(x,\partial\Omega) > R^*\}. \tag{1.6}$$

The approach of [134] to the proof of Theorem 1.1 is variational, involving the construction of various judicious radial comparison functions, see also [12]. Although variational, in our opinion, their argument boils down to the construction of a weak lower solution to (1.2), see [35], whose building blocks, after a translation, are radial solutions of

$$\Delta\Phi^r + c^2(\mu - \Phi^r) = 0 \text{ in } B_r, \ \Phi^r(r) = \mu - a; \ -\Delta\Psi^{r,R} = 0 \text{ in } B_{r+R}\backslash B_r, \ \Psi^{r,R}(r) = \mu - a, \ \Psi^{r,R}(r+R) = 0, \tag{1.7}$$

where $c^2 < W''(t)$, $t \in [\mu - a, \mu]$ (note that assumption **(b)** implies that solutions of (1.2) are super-harmonic). It can be verified that $\Phi'(r) < \Psi'(r)$ for sufficiently large $r$ and $R$ (having dropped the superscripts for convenience). So, after a translation, the functions $u$, $v$, and zero, can be patched together at $|x| = r$ and $|x| = r + R$ to form a weak lower solution to

(1.2), in the sense of [35], provided that $\Omega$ contains some large ball of radius greater than $r+R$. This gives us a solution satisfying (1.4) only in $B_r$ (we use $\mu$ as an upper solution). However, we may extend the domain of validity, and obtain the desired bound (1.4), by "sliding around" that lower-solution, as in [38]. Using this strategy, one may considerably simplify the corresponding arguments in [134]. We note that, once (1.4) is established, the proof of the exponential decay estimate (1.5), given in [134], can be simplified considerably by employing Lemma 4.2 in [129], making use of the non-degeneracy condition **(c)** (the constants in Theorem 1.1 can be chosen so that $W''(t) > 0,\ t \in [\mu - a^*, \mu]$). Moreover, an examination of the proof of Lemma 2.1 in [134] (see Lemma A.1 herein) shows that assumption **(a)** above can be relaxed to

**(a'):** There exists a constant $\mu > 0$ such that

$$0 = W(\mu) < W(t),\ t \in [0,\mu),\quad W(t) \geq 0,\ t \in \mathbb{R},$$
$$W(-t) \geq W(t),\ t \in [0,\mu] \text{ or } W'(t) < 0,\ t < 0.$$

For a typical example of such a potential, see Figure 1.1.

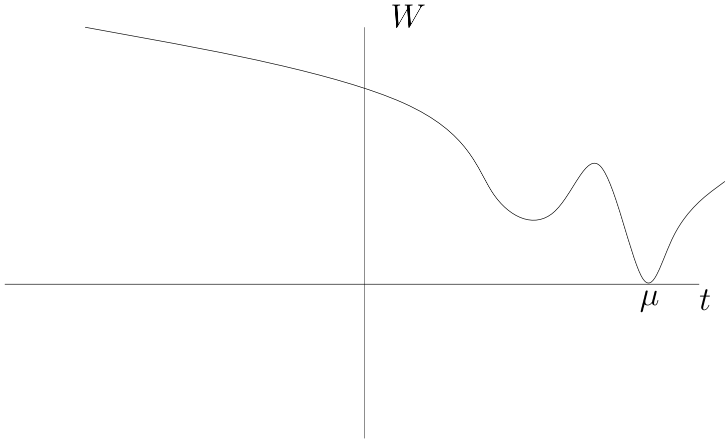


FIGURE 1.1. An example of a potential $W$ satisfying hypothesis **(a')**.

If one further assumes that

$$W''(0) < 0 \ \text{ if } \ W'(0) = 0, \tag{1.8}$$

and

$$W'(t) < 0, \quad t \in (0,\mu), \tag{1.9}$$

then Theorem 1.1 can essentially be deduced from Lemmas 3.2–3.3 in the famous article [38] by Berestycki, Caffarelli and Nirenberg or Lemma 4.1 in the recent article [172] by Pacard, Kowalczyk and Liu, see also Lemmas 6.1–6.2 in [224], and [137]. In fact, the latter lemmas hold for arbitrary positive solutions to (1.2) with values less than $\mu$.

The main purpose of this article is to show that relation (1.4) can be established in a simple manner *without* assuming the monotonicity condition **(b)**, and in fact we will prove a stronger version of it. A well known nonlinearity which satisfies our assumptions but not **(b)** is

$$W'(u) = u(u-a)(u-\mu) \text{ with } 0 < a < \frac{\mu}{2}, \tag{1.10}$$

which arises in the mathematical study of population genetics (see [28]). Moreover, we remove completely the non-degeneracy condition **(c)** from the proof of (1.4). On the other hand, since an argument of [134] involving the boundary regularity of weak solutions to (1.2) when $\partial\Omega$ is arbitrarily Lipschitz is not clear to us (see the last part of the proof of Theorem 3.3 therein), we will assume that $\Omega$ has $C^2$-boundary (to be on the safe side, see however Remarks 1.4 and 2.16 that follow). We will accomplish the aforementioned improvements, loosely speaking, by using translations of a positive solution of

$$\Delta u = W'(u), \ x \in B_R; \quad u(x) = 0, \ x \in \partial B_R,$$

which minimizes the associated energy, as a lower solution of (1.2) after we have extended it by zero outside of $B_R$. Actually, this approach will allow us to refine the results of [38], [172] that we mentioned earlier in relation to (1.8), (1.9). On the other side, assuming further that $W'$ satisfies a scaling property and that the corresponding whole space problem (1.22) below does not have nontrivial entire solutions (a Liouville type theorem), we will use "blow-up" arguments from [141] together with a key "doubling lemma" from [206] to establish that Lemma 3.3 in [38] can be improved.

In passing, we remark that a similar monotonicity assumption to **(b)** also appears in a series of papers [12], [13], [16] in the context of variational elliptic systems of the form $\Delta u = \nabla_u W(u)$ with $W : \mathbb{R}^n \to \mathbb{R}$. In particular, these references employ comparison functions of the form (1.7). In this direction, see also Remarks 1.5, 2.9 and Appendix D below.

Our main result is

**Theorem 1.2.** Assume that $\Omega$ is a domain with nonempty boundary of class $C^2$, and that $W \in C^2$ satisfies **(a')**. Let $\epsilon \in (0, \mu)$ and $D > D'$, where $D'$ is determined from the relation

$$\mathbf{U}(D') = \mu - \epsilon, \tag{1.11}$$

where in turn $\mathbf{U}$ is the only function in $C^2[0,\infty)$ that satisfies

$$\mathbf{U}'' = W'(\mathbf{U}), \ s > 0; \quad \mathbf{U}(0) = 0, \ \lim_{s\to\infty} \mathbf{U}(s) = \mu, \tag{1.12}$$

(see Remark 1.1 below). There exists an $R' > D$, depending only on $\epsilon$, $D$, $W$, and $n$, such that if $\Omega$ contains some closed ball of radius $R'$ then problem (1.2) has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ verifying (1.3), and

$$\mu - \epsilon \le u(x), \quad x \in \Omega_{R'} + B_{(R'-D)}, \tag{1.13}$$

where $\Omega_{R'}$ was previously defined in (1.6). Furthermore, it holds that

$$\min\{W(t) \ : \ t \in [0, u(x)]\} \le \frac{C}{\mathrm{dist}(x, \partial\Omega)}, \quad x \in \Omega_{R'}, \tag{1.14}$$

for some constant $C > 0$ that depends only on $W, n$.
If $W''(\mu) > 0$ then estimate (1.5) holds true.
If

$$W''(t) \ge 0 \ \text{ for } \ \mu - t > 0 \ \text{ small}, \tag{1.15}$$

then

$$-W'(u(x)) \le \frac{\tilde{C}}{(\mathrm{dist}(x, \partial\Omega))^2}, \quad x \in \Omega_{R'}, \quad R \ge R', \tag{1.16}$$

for some constant $\tilde{C} > 0$ that depends only on $n$, assuming that $W'' \geq 0$ on $[\mu - \epsilon, \mu]$.
If there exist constants $c > 0$ and $p > 1$ such that

$$-W'(t) \geq c(\mu - t)^p, \quad t \in [\mu - d, \mu], \quad \text{for some small } d > 0, \tag{1.17}$$

and $\bar{\Omega}$ is disjoint from the closure of an infinite open connected cone, or $n = 2$ and $\bar{\Omega} \neq \mathbb{R}^2$, then

$$\mu - u \leq \tilde{K} \text{dist}^{-\frac{2}{p-1}}(x, \partial\Omega), \quad x \in \Omega, \tag{1.18}$$

for some constant $\tilde{K} > 0$ that depends only on $c$, $p$, $n$ and $W$.

Estimate (1.16) is motivated from [38]. Condition (1.17) is in part motivated by some recent studies [242, 243, 244] of a class of singularly perturbed elliptic boundary value problems of the form (8.5) below in *one space dimension*, where the degenerate equation $W(u, x) = 0$ has a root $u = u_0(x)$ of finite multiplicity.

The method of our proof is quite flexible, and we came up with a variety of applications to related problems that can be found in the following sections and the included remarks (see the outline at the end of this section). As will be apparent from the proof, see in particular the comments leading to Proposition 8.1 below, a delicacy of our result is that the constant $D'$ is independent of $n$.

**Remark 1.1.** The existence and uniqueness of such a solution $\mathbf{U}$ of the ordinary differential equation $u'' = W'(u)$ follows readily from **(a')** by phase plane analysis, using the fact that the latter equation has the conserved quantity $e(s) = \frac{1}{2}(u')^2 - W(u)$, see for instance Lemma 3.2 in [22], Chapter 2 in [27] or page 135 in [247] (for a more analytic approach, we refer to [36] or [54]). We note that

$$\mathbf{U}'(s) > 0, \quad s \geq 0. \tag{1.19}$$

**Remark 1.2.** Similar assertions hold for the Robin boundary value problem:

$$\Delta u = W'(u), \ x \in \Omega; \ \ \frac{\partial u}{\partial \nu} + b(x)u = 0, \ x \in \partial\Omega,$$

where $\nu$ denotes the outward unit normal vector to the boundary of $\Omega$, assuming here that the latter is at least $C^1$, with $b \in C^{1+\alpha}(\partial\Omega)$, $\alpha > 0$, being nonnegative (so that the constant $\mu$ is a positive upper solution, see [217]). Moreover, as in [134], we can study some problems with mixed boundary conditions (see also Section 9).

**Remark 1.3.** A sufficient, and easy to check, condition for the uniqueness of a positive solution of (1.2), in *any smooth bounded domain*, is

$$\frac{W'(t)}{t} \ \text{being strictly increasing in} \ \ (0, \infty). \tag{1.20}$$

This uniqueness result is originally due to Krasnoselski, see [59] (see also Proposition 3.5 in [194] or Theorem 1.16 in [214] for a different proof, and Theorem 3 in [231] for a radially symmetric proof). The above condition is clearly satisfied by the model double well potential in (1.23) below. Related conditions can be found in [240]. In certain cases, these type of conditions imply uniqueness of a positive solution in unbounded domains as well, see for example [72] and [99] for uniqueness of the so-called saddle solutions that we will discuss shortly. Another sufficient condition, which on the other hand depends partly on the smooth bounded domain $\Omega$, is

$$W''(t) \geq -\lambda, \quad t \geq 0,$$

for some $\lambda < \lambda_1$, where $\lambda_1 > 0$ denotes the principal eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega)$ (see [21], [223]). This condition is clearly satisfied, with $\lambda = 0$, by the convex model potential in (4.1) below.

Let us mention that for a class of potentials, including (1.23), the dependence of the set of solutions of (1.2), in one space dimension, on the size of the interval was studied for the first time in [82] (see also the more up to date reference [84]).

In our opinion, Theorems 1.1 and 1.2 are important for the following reasons. If we additionally assume that $W$ is even, namely

$$W(-t) = W(t), \quad t \in \mathbb{R}, \tag{1.21}$$

by means of these theorems, we can derive the existence of various sign-changing entire solutions for the problem

$$\Delta u = W'(u), \quad x \in \mathbb{R}^n. \tag{1.22}$$

This can be done by first establishing existence of a positive solution in a suitable large "fundamental" domain $\Omega_F \subset \mathbb{R}^n$, with Dirichlet boundary conditions on $\partial\Omega_F$, and then performing consecutive odd reflections to cover the entire space.

**Remark 1.4.** The boundary of the fundamental domain $\Omega_F$ may have corner or conical points. But we can round them off, approximating $\Omega_F$ by a sequence of expanding *smooth* domains $\Omega_j$ (where Theorem 1.2 is applicable). Then, we can obtain the desired solution in $\Omega_F$ by letting $j \to \infty$ along a subsequence (see [99]). In this regard, see also Remark 2.16 below.

The fact that, after reflecting, we obtain a classical solution can be shown by a standard capacity argument (see Theorem 1.4 in [69]).

In the case where

$$W(t) = \frac{1}{4}(t^2 - 1)^2, \quad t \in \mathbb{R}, \tag{1.23}$$

then (1.22) becomes the well known Allen-Cahn equation (see for instance [203]). Assuming that $W$ is even, namely that (1.21) holds true, then (1.2) has always the trivial solution. In this regard, the purpose of estimate (1.13) is twofold: In the case where $\Omega_F$ is bounded, it ensures that the solution of (1.2) (on $\Omega_F$), provided by Theorem 1.2, is nontrivial. The situation of unbounded domains $\Omega_F$ can be treated by exhausting them by an increasing (with respect to inclusions) sequence $\{\Omega_j\}$ of bounded ones, each containing the same ball $B_{R'}(x_0)$, and a standard compactness argument, making use of (1.3) together with elliptic estimates and a Cantor type diagonal argument. The fact that the region of validity of estimate (1.13) increases, as $j \to \infty$, rules out the possibility of subsequences of the (chosen) solutions $u_j$ of $(1.2)_j$ on $\Omega_j$ converging, uniformly in compact subsets of $\Omega_F$, to the trivial solution of (1.2) on $\Omega_F$. Another approach for excluding this last scenario, which however does not seem to provide uniform estimates directly, can be found in the proof of Theorem 1.3 in [69], based on a similar relation to (2.71) below (see also [104] and [203]). In this fashion, and under more general assumptions on $W$ than previous studies (conditions **(a')** and (1.21) suffice for most applications), one can construct a whole gallery of nontrivial sign-changing solutions of (1.22) that includes

- "saddle solutions" which vanish on the Simons cone $\{(x, y) \in \mathbb{R}^{2m} : |x| = |y|\} \subset \mathbb{R}^{2m} = \mathbb{R}^n$ if $n$ is even (see [69], [71], [72], [99], [144], and [203]). In fact, they can be constructed in the *block-radial* class, namely $u(x, y) = \mathrm{u}(|x|, |y|) = -\mathrm{u}(|y|, |x|)$. In passing, we note that solutions with these symmetries have been studied for nonlinear Schrödinger type equations, say (1.22) with $W'(t) = t - t^3$, in Chapter 3 in [174], Section 1.6 in [252], and the references therein (for such solutions to the Gross-Pitaevskii equation with radial trapping potential, we refer to Section 6 in [161]). Estimate (1.5) implies that the corresponding saddle solution converges to $\pm\mu$ exponentially fast, as the signed distance from the Simons cone tends to plus/minus infinity respectively. Analogous solutions exist in odd dimensions, for example when $n = 3$ it was shown in [5] that there exists a solution which vanishes on all coordinate planes (see also a related discussion in [105]). In dimension $n = 2$, solutions whose zero level set has the symmetry of a regular $2k$–polygon and consists of $k$ straight lines passing through the origin were found in [4] (in the case where $W$ is periodic, similar solutions but with polynomial growth were found, following this strategy, recently in [248]); such solutions can appropriately be named "pizza solutions", see also [226]. Denote $G$ the rotation of order $2k$, and note that these solutions satisfy $u(Gx) = -u(x)$, $x \in \mathbb{R}^2$. Another method to get $u$ is to find a minimizing solution $u_R$ of the equation in the invariant class $\{u \in W_0^{1,2}(B_R) \text{ and } u(Gx) = -u(x),\ x \in B_R\}$. The minimizer $u_R$ can be proved to satisfy (1.2) in $B_R$ by the heat flow method (see [12], [11], [16], [33], [46]). Note that because $W$ is even, the invariant class is positively invariant by the heat flow.

- "lattice solutions" which include solutions that are periodic in each variable $x_i$ with period $L_i$, provided that $L_i$, $i = 1, \cdots, n$, are sufficiently large (see [16], [34], [130], [164], and [189]). This type of solutions, which can be described as having lamellar phase, were recently conjectured to exist in Chapter 4 of [230]. Another example, which is motivated from [187], are solutions in the plane whose nodal domains consist of sufficiently large (identical modulo translation and rotation) equilateral triangles tiling the plane (in relation to this, see also Remark 2.20 below). Under some additional hypotheses on $W$, planar lattice solutions can be constructed by local and global bifurcation techniques (see [130], [154], [164], and [187]).
- "tick saddle solutions" which have saddle (or pizza) structure in some coordinates while they are periodic in the remaining ones (see the introduction in [134]). For example, in $\mathbb{R}^2$, these solutions are odd with respect to both $x$ and $y$, having as nodal curves the lines $x = 0$ and $y = kL$, $k \in \mathbb{Z}$, for $L$ sufficiently large (so that the fundamental domain $\Omega_{F,L} \equiv \{x > 0,\ y \in (0, L)\}$ contains a sufficiently large closed ball). In fact, if $W''(0) < 0$ and (2.43) below hold, by modifying the approach of the current paper and using some ideas from Proposition 3.1 in [104] (which dealt with a problem of similar nature on an infinite half strip, see also Remark 2.10 below), it is plausible that there exists an explicit constant $L^* > 0$ such that (1.2) considered in $\Omega_{F,L}$ has a positive solution if and only if $L > L^*$ (see also Remark 2.6 below); a similar construction should also work in higher dimensions. We note that tick saddle solutions can be constructed as limits of appropriate lattice solutions by letting some of the periods tend to infinity (along a subsequence), see [16]. In the case where $W$ is as in (1.23), and $n = 2$, the spectrum of the linearized operator about the saddle solution of [99] has a unique negative eigenvalue (see [220]). Moreover, it has been shown recently that the saddle solution is non-degenerate, namely there are no decaying elements in the kernel of the linearized operator (see [171]). In view of these two properties it might also be possible to construct tick saddle solutions in

$\mathbb{R}^3$, with $W$ as in (1.23), by local bifurcation techniques (for example, by the ideas in [95]). Lastly, let us point out that the shape of their zero set bears some qualitative similarities to Sherk's singly periodic minimal surface (see for example [88]).

- "Screw–motion invariant solutions" whose nodal set is a helicoïd of $\mathbb{R}^3$, or analogous minimal surfaces in any odd dimension (see [104] and Remark 2.10 herein).

A completely different approach to the construction of sign-changing solutions of (1.22), mainly applied for potentials satisfying **(a)**, **(c)**, and (1.21) (the typical representative being (1.23)), is based on the implementation of an infinite dimensional Lyapunov–Schmidt reduction argument, see [102], [104], [105], [203], and the references therein. This approach produces solutions with less (or even without any) symmetry but is technically more involved.

Our Theorem 1.2 can also be used to construct multiple positive solutions of (1.2), using estimate (1.13) to make sure that they are distinct, see Section 7 below.

1.1. **Outline of the paper.** The outline of the paper is as follows: In Section 2, we will present the proof of our main result, with the exception of (1.18), by using two different approaches, both based on a special case of a radial lemma that we prove in Subsection 2.1. In the remainder of the paper we will exploit further this radial lemma and use it as a basis to prove interesting results. In Section 3, we prove uniform lower bounds for arbitrary positive solutions. In Section 4, we prove universal decay estimates for solutions, in the case where $W$ is a model power nonlinearity potential, thereby generalizing the exponential decay estimate (1.5) by an algebraic one and relating the obtained result to a corresponding one in [38]. Moreover, this algebraic decay estimate allows us to show (1.18) and thus complete the proof of Theorem 1.2. In Section 5, under appropriate conditions on $W$, we will show that all entire solutions of (1.22) are uniformly bounded; combining this with the main result of Section 3, we can give a short self-contained proof of the main result in the paper of Du and Ma [108]. In Section 6, we prove nonexistence results for nonconstant solutions with Neumann boundary conditions that are motivated by some Liouville type result of Berestycki, Hamel and Matano [43] (for which we provide simplified proofs, while at the same time removing a technical assumption). In Section 7, we will show how our Theorem 1.2 can be used to produce multiple positive solutions of (1.2) and thus generalize an old result of P. Hess from 1981, where nonlinear eigenvalue problems were considered. In Section 8, we study the size of the boundary layer of global minimizers of the corresponding singular perturbation problem, in the context of nonlinear eigenvalue problems. In Section 9, we will study the corresponding problem with mixed boundary conditions. In Section 10, we will prove some new one-dimensional symmetry results for certain entire solutions to (1.22), by exploiting for the first time an old result of Caffarelli, Garofalo, and Segála [74], and we suggest a connection with the theory of minimal surfaces. Exploiting this approach, and the Hamiltonian structure of the equation, we can give a new proof of Gibbons' conjecture, originally proven independently by [31, 42, 118] (see also [76]). This conjecture is a weak form of the famous conjecture of De Giorgi. Furthermore, we are able to provide new proofs of well known symmetry results in half-spaces with Dirichlet boundary conditions. Moreover, we generalize a rigidity result of [120]. Finally, in Section 11, we study the one-dimensional symmetry of solutions in convex cylindrical domains with Neumann boundary conditions. In Appendix A, for completeness purposes, we will state some useful comparison lemmas that we will use in this article. In Appendix B, for the reader's convenience, we will state a useful Liouville type theorem of [121] which extends a result of [58]. In Appendix C, for the reader's convenience, we will state the useful doubling lemma of [206] that we mentioned earlier. In Appendix D, we make some remarks that are motivated from the recent paper [13], dealing with uniform estimates for equivariant entire solutions to an elliptic system under assumptions that are analogous to those in [134].

**Remark 1.5.** We recently found the paper [16], where it is stated that G. Fusco, in work in progress (now published, see [135], and also [136]), has been able to remove the corresponding monotonicity assumption to **(b)** from the vector-valued Allen-Cahn type equation that was treated in [12]. After the first version of the current paper was completed, we were informed by G. Fusco that himself, F. Leonetti and C. Pignotti are working in a paper where, using the same technique developed for the vector case, they are in the process of extending the main result in [134] to more general potentials without assuming **(b)**. Their approach is certainly more elaborate but it is entirely self-contained, while we use in a simple and coordinate way various deep well known results.

## 2. Proof of the main result

2.1. **Minimizers of the energy functional on large balls.** In this subsection, we will mainly prove two lemmas concerning the asymptotic behavior of the minimizing (of the associated energy) solutions of (1.2) over large balls as their radius tends to infinity. The first one is essential for the proof of Theorem 1.2, and refines a result of P. Clément and G. Sweers [86]. The latter result is quite useful, and has been previously applied in singular perturbation problems (see [96], [173], and [179]). The second lemma, an extension of the first, is of independent interest and in particular allows for $W'(0)$ to be positive. Even though the first lemma is a special case of the second, we felt that it would be more instructive and more convenient for the reader to present them separately, since the more general second lemma is not needed for the proof of Theorem 1.2 and can be skipped at first reading.

The following is our first lemma, which is motivated from Lemma 2 in [173] and Lemma 2.2 in [179] (see also Lemma 2.4 in [114]), whose origins can be traced back to [85, 86]. In these works, the weaker relation (2.12) below was established, which implies that assertion (2.3) holds *but* with constant $D$ possibly *diverging* as $n \to \infty$ (see also Remark 2.2 below). Our improvement turns out to have interesting consequences in the study of the boundary layer of solutions of singular perturbation problems of the form (7.9) below, with $\lambda = \varepsilon^{-1} \to \infty$, see Remark 7.2 and Section 8 below. Moreover, estimate (2.3) will be used in a crucial way in Proposition 2.1 for studying the asymptotic stability of minimizing solutions that are provided by the following lemma or the more general Lemma 2.3 below.

**Lemma 2.1.** Assume that $W \in C^2$ satisfies condition **(a')**. Let $\epsilon \in (0, \mu)$ and $D > D'$, where $D'$ is as in (1.11). There exists a positive constant $R' > D$, depending only on $\epsilon$, $D$, $W$ and $n$, such that there exists a global minimizer $u_R$ of the energy functional

$$J(v; B_R) = \int_{B_R} \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx, \quad v \in W_0^{1,2}(B_R), \tag{2.1}$$

which satisfies

$$0 < u_R(x) < \mu, \ x \in B_R, \tag{2.2}$$

and

$$\mu - \epsilon \leq u_R(x), \ x \in \bar{B}_{(R-D)}, \tag{2.3}$$

provided that $R \geq R'$. Moreover, there exists a constant $C$ depending only on $W, n$ such that

$$\min\{W(t)\ :\ t \in [0, u_R(r)]\} \leq \frac{C}{R-r}, \quad r \in [0, R),\ \forall\ R \geq R'. \tag{2.4}$$

(If necessary, we assume that $W$ is extended linearly outside of a large compact interval so that the above functional is well defined (see also Lemma 2.4 in [114]); clearly this modification does not affect the assertions of the lemma).

*Proof.* Under our assumptions on $W$, it is standard to show the existence of a global minimizer $u_R \in W_0^{1,2}(B_R)$ satisfying

$$0 \leq u_R(x) \leq \mu \quad \text{a.e. in } B_R, \tag{2.5}$$

see Chapter 2 in [29], [134], [213], and Lemma A.1 herein (applied to the minimizing sequence converging, weakly in $W_0^{1,2}(B_R)$, to $u_R$). (The upper bound in (2.5) can also be derived from Lemma A.3 below, see also the second proof of Theorem 1.2). By standard elliptic regularity theory [142], this minimizer is a smooth solution, in $C^2(\bar{B}_R)$, of

$$\Delta u = W'(u) \quad \text{in} \quad B_R; \ u = 0 \quad \text{on} \quad \partial B_R. \tag{2.6}$$

By the strong maximum principle (see for example Lemma 3.4 in [142]), via (2.5) and (2.6), we deduce that $u_R(x) < \mu,\ x \in B_R$, and that either $u_R$ is identically equal to zero or $u_R(x) > 0,\ x \in B_R$ (recall that assumption **(a')** implies that $W'(0) \leq 0$ and $W'(\mu) = 0$).

By adapting an argument from Section 4 in [203] (see also Lemma 5.3 in [138] and Theorem 1.13 in [213]), we will show that $u_R$ is nontrivial, provided that $R$ is sufficiently large (depending only on $W$ and $n$). (This is certainly the case when $W'(0) < 0$). It is easy to cook up a test function, and use it as a competitor, to show that there exists a positive constant $C_1$, depending only on $W$ and $n$, such that

$$J(u_R; B_R) \leq C_1 R^{n-1}, \quad \text{say for} \quad R \geq 2. \tag{2.7}$$

(Plainly construct a function which interpolates smoothly from $\mu$ to 0 in a layer of size 1 around the boundary of $B_R$ and which is identically equal to $\mu$ elsewhere, see also (2.70) below or Lemma 1 in [75]). In fact, as in Proposition 1 in [2] (see also [175]), it can be shown that

$$J(u_R; B_K) \leq \tilde{C}_1 K^{n-1} \quad \forall\ K < R, \quad R \geq 2, \tag{2.8}$$

where the constant $\tilde{C}_1 > 0$ depends only on $W$ and $n$ (see also Remark 2.11, and the arguments leading to relation (2.71) below). On the other hand, the energy of the trivial solution is

$$J(0; B_R) = \int_{B_R} W(0)dx = C_2 R^n,$$

where $C_2 > 0$ depends only on $W,\ n$. From (2.7), and the above relation, we infer that $u_R$ is certainly not identically equal to zero for

$$R \geq C_1 C_2^{-1} + 2.$$

We thus conclude that (2.2) holds. (In the above calculation, we relied on the fact that **(a')** implies that $W(0) > 0$; in this regard, see Remark 2.8 below).

Since $u_R \in C^2(\bar{B}_R)$ is strictly positive in the ball $B_R$, by (2.6) and the method of moving planes [61, 92, 140], we infer that $u_R$ is radially symmetric and decreasing, namely

$$u_R'(r) < 0, \quad r \in (0, R), \tag{2.9}$$

(with the obvious notation). In this regard, keep in mind that if $v \in W_0^{1,2}(B_R)$ is nonnegative, then its Schwarz symmetrization $v^* \in W_0^{1,2}(B_R)$, which is radially symmetric and decreasing, satisfies $J(v^*; B_R) \leq J(v; B_R)$ (see for example [64] and the references therein). We note that, since $u_R$ is a global minimizer and thus stable (in the usual sense, as described in Remark 2.17 below), the radial symmetry of $u_R$, for $n \geq 2$, can also be deduced as in Lemma 1.1 in [9] (see also the related references in the proof of Lemma 2.3 below). In fact, the monotonicity property (2.9) can be alternatively derived by arguing as in Lemma 2 in [68] (see also Proposition 1.3.4 in [113]), making use of the stability of the radial solution $u_R$ (see also the proof of Lemma 2.3 below, and Lemma 1 in [8]). Now, relation (2.7) and the nonnegativity of $W$ clearly imply that

$$\int_{B_R \setminus B_{\frac{R}{2}}} \left\{ \frac{1}{2}|\nabla u_R|^2 + W(u_R) \right\} dx \leq C_1 R^{n-1}, \quad R \geq C_1 C_2^{-1} + 2. \tag{2.10}$$

Hence, by the mean value theorem and the radial symmetry of $u_R$, there exists a $\xi \in \left(\frac{R}{2}, R\right)$ such that

$$\left\{ \frac{1}{2}[u_R'(\xi)]^2 + W\left(u_R(\xi)\right) \right\} \left| B_R \setminus B_{\frac{R}{2}} \right| \leq C_1 R^{n-1}, \quad R \geq C_1 C_2^{-1} + 2,$$

i.e.,

$$\frac{1}{2}[u_R'(\xi)]^2 + W\left(u_R(\xi)\right) \leq C_3 R^{-1}, \quad R \geq C_1 C_2^{-1} + 2, \tag{2.11}$$

where the positive constat $C_3$ depends only on $W$ and $n$ (for simplicity in notation, we have suppressed the obvious dependence of $\xi$ on $R$). Hence, from assumption **(a')**, and relations (2.9), (2.11), we obtain that

$$u_R \to \mu, \text{ uniformly in } \bar{B}_{\frac{R}{2}}, \text{ as } R \to \infty. \tag{2.12}$$

In the sequel, we will prove that the stronger property (2.3) holds true.

For future reference, we note here that

$$[u_R'(R)]^2 \to 2W(0) \quad \text{as} \quad R \to \infty. \tag{2.13}$$

Indeed, let

$$E_R(r) = \frac{1}{2}[u_R'(r)]^2 - W\left(u_R(r)\right), \quad r \in (0, R). \tag{2.14}$$

Thanks to (2.6), we find that

$$E_R'(r) = u_R'' u_R' - W'(u_R) u_R' = -\frac{n-1}{r}(u_R')^2, \quad r \in (0, R). \tag{2.15}$$

So,

$$E_R(R) = E_R(\xi) - \int_\xi^R \frac{n-1}{r}(u_R')^2 dr, \tag{2.16}$$

where $\xi \in \left(\frac{R}{2}, R\right)$ is as in (2.11). Now, observe that (2.10) and the nonnegativity of $W$ imply that

$$\int_\xi^R r^{n-1}(u_R')^2 dr \leq C_4 R^{n-1}, \quad R \geq C_1 C_2^{-1} + 2,$$

with $C_4$ depending only on $W$ and $n$. In turn, the above estimate clearly implies that

$$\int_\xi^R (u_R')^2 dr \leq 2^{n-1} C_4, \quad R \geq C_1 C_2^{-1} + 2,$$

and it follows that

$$\int_{\xi}^{R} \frac{n-1}{r}(u_R')^2 dr \leq 2^n C_4 (n-1) R^{-1}, \quad R \geq C_1 C_2^{-1} + 2. \tag{2.17}$$

The claimed relation (2.13) follows readily from (2.11), (2.14), (2.16), and (2.17). In fact, we have shown that $R|E_R(R)|$ remains uniformly bounded as $R \to \infty$. In relation to (2.13), see also Remark 8.5 below.

We also consider the following family of functions

$$U_R(s) = u_R(R-s), \quad s \in [0, R]. \tag{2.18}$$

We claim that

$$U_R \to \mathbf{U}, \text{ uniformly on compact intervals of } [0,\infty), \text{ as } R \to \infty, \tag{2.19}$$

where $\mathbf{U}$ is as in (1.12).

In view of (2.6), we get

$$U_R'' - \frac{n-1}{R-s} U_R' - W'(U_R) = 0, \quad s \in (0, R). \tag{2.20}$$

Making use of (2.2), the above equation, elliptic estimates [142], Arczela-Ascoli's theorem, and a standard diagonal argument, passing to a subsequence $R_i \to \infty$, we find that

$$U_{R_i} \to V \text{ and } U_{R_i}' \to V', \text{ uniformly on compact intervals of } [0,\infty), \text{ as } i \to \infty, \tag{2.21}$$

where $V \in C^2[0,\infty)$ is nonnegative and satisfies

$$V'' = W'(V), \ s > 0, \text{ and } V(0) = 0. \tag{2.22}$$

Moreover, by (2.13), (2.18), and (2.21), we see that

$$[V'(0)]^2 = 2W(0) > 0.$$

By the uniqueness of solutions of initial value problems of ordinary differential equations, see for example page 108 in [247], we deduce that

$$V \equiv \mathbf{U},$$

where $\mathbf{U}$ is as in (1.12). We also used that $\mathbf{U}$, $V$ are nonnegative (which implies that $\mathbf{U}'(0)$, $V'(0)$ are also nonnegative), and the relation

$$[\mathbf{U}'(0)]^2 = 2W(0), \tag{2.23}$$

which follows from the identity

$$[\mathbf{U}'(s)]^2 - [\mathbf{U}'(0)]^2 = 2\int_0^s W'(\mathbf{U})\mathbf{U}' ds = 2W(\mathbf{U}(s)) - 2W(0), \quad s \geq 0,$$

and the fact that $\mathbf{U}(s) \to \mu$ as $s \to \infty$, recalling that $W(\mu) = 0$ (otherwise, $\mathbf{U}'(s)$ would tend to a nonzero number and in turn $|\mathbf{U}(s)|$ would diverge, as $s \to \infty$). Moreover, by the uniqueness of the limiting function, we infer that the limits in (2.21) hold for *all* $R \to \infty$. Consequently, the claimed relation (2.19) holds.

Having (2.13), (2.19) at our disposal, we can now proceed to the proof of (2.3). Let $\epsilon \in (0, \mu)$ and $D > D'$, where $D'$ is as in (1.11). By virtue of (1.12), (1.19), and (2.19), there exists a sufficiently large $R'$, depending only on $\epsilon$, $D$, $W$, $n$, such that $U_R(D) \geq \mu - \epsilon$, and all the previous relations continue to hold, for $R > R'$. In other words, via (2.18), we have that

$$u_R(R - D) = U_R(D) \geq \mu - \epsilon, \quad R > R'. \tag{2.24}$$

The fact that $u_R$ is radially decreasing, recall (2.9), and the above relation imply the validity of (2.3). As will be apparent from the Remarks 2.3 and 2.4 that follow, condition (2.9) is essential only when dealing with degenerate situations when there exists a sequence $t_j \to \mu^-$ such that $W'(t_{2j})W'(t_{2j+1}) < 0$ for large $j$; an example is a potential $W$ that coincides with $(\mu - t)^2 \left[\sin\left(\frac{1}{\mu-t}\right) + 2\right]$ near $\mu$, in which case we can choose $t_j = \mu - \frac{1}{j\pi}$ (note that $W'(t) \sim \cos\left(\frac{1}{\mu-t}\right)$ as $t \to \mu^-$). It remains to prove (2.4). To this end, note that the nonnegativity of $W$ and (2.7) imply that

$$\int_r^R s^{n-1} W\left(u_R(s)\right) ds \leq \tilde{C}_1 R^{n-1}, \quad r \in (0, R),$$

where $\tilde{C}_1$ is independent of $R \geq R'$. It follows, via (2.9), that

$$\min\left\{W(t) \ : \ t \in [0, u_R(r)]\right\}(R^n - r^n) \leq n\tilde{C}_1 R^{n-1},$$

which clearly implies the validity of (2.4).

The proof of the lemma is complete. □

**Remark 2.1.** Our assumptions on the behavior of $W$ near its global minimum at $\mu$ are quite weak, and in fact even allow for the potential $W$ to have $C^\infty$ contact with zero at the point $\mu$, that is $W^{(i)}(\mu) = 0, \ i \geq 1$. This degeneracy translates into the absence of decay rates for the convergence of the "inner" approximate solution $\mathbf{U}(R - |x|)$ (in the sense of singular perturbation theory, see [129] and the related references that can be found in Remark 8.6 below), where $\mathbf{U}$ is as described in (1.12), to the "outer" one $\mu$, away from the boundary of $B_R$, as $R \to \infty$ (see also the discussion leading to (7.10) below). This is the main reason why we have not attempted to apply a perturbation argument, see for instance [129] and the related references in Remark 8.6 below, in order to study the asymptotic behavior of $u_R$ as $R \to \infty$. We refer to the recent papers [242, 243, 244] for singular perturbation arguments (in one space dimension) in the case where $\mu$ is a root of $W'$ of finite multiplicity (also allowing for $x$ dependence on $W'$). From the viewpoint of geometric singular perturbation theory, the case $W''(\mu) = 0$ corresponds to lack of normal hyperbolicity of the slow manifold corresponding to the equilibria with $(u, u') = (\mu, 0)$ (see [241]).

If $W''(\mu) > 0$, then the convergence of $\mathbf{U}$ to $\mu$ is of order $e^{-\sqrt{W''(\mu)}s}$ as $s \to \infty$ (by the stable manifold theorem, see [87]), and one can effectively interpolate between the outer and inner approximations in order to construct a smooth global approximation that is valid throughout $B_R$.

**Remark 2.2.** By the well known relations $|B_R| = c_n R^n$, $|\partial B_R| = nc_n R^{n-1}$, $R > 0$, $n \geq 2$, for some explicit constants $c_n$ (independent of $R$), where $|\partial B_R|$ denotes the $(n-1)$–dimensional measure of $\partial B_R$, we find that

$$\frac{|\partial B_R|}{\left|B_R \backslash B_{\frac{R}{2}}\right|} = \frac{n2^n}{2^n - 1} R^{-1}, \quad R > 0.$$

We deduce that the constant $R'$ in Lemma 2.1 diverges (at least linearly) as $n \to \infty$ (see in particular the relations leading to (2.11)).

**Remark 2.3.** If in addition to **(a')** we assume that there exists some $d \in (0, \mu)$ such that

$$W'(t) \leq 0, \quad t \in (\mu - d, \mu), \tag{2.25}$$

(note that this is very natural), then relation (2.3) can alternatively be shown, starting from (2.24), *without* assuming knowledge of (2.9), as follows: Assuming, without loss of generality, that $2\epsilon < d$, thanks to Lemma A.2 below, we can find a radial $\tilde{u} \in W^{1,2}(B_{R-D})$ such that

$$J(\tilde{u}; B_{R-D}) \leq J(u_R; B_{R-D}), \quad \tilde{u}(R-D) = u_R(R-D), \quad \text{and} \quad \tilde{u}(x) \in [\mu - \epsilon, \mu], \ x \in \bar{B}_{R-D}.$$

Thus, the function

$$\hat{u}(x) = \begin{cases} \tilde{u}(x), & x \in B_{R-D}, \\ \\ u_R(x), & x \in B_R \backslash B_{R-D}, \end{cases}$$

belongs in $W_0^{1,2}(B_R)$ and is a global minimizer of $J(\cdot; B_R)$ in $W_0^{1,2}(B_R)$ (since $J(\hat{u}; B_R) \leq J(u_R; B_R)$). In particular, it is smooth, radial (and by virtue of its construction), and solves (2.6). It follows from Lemma 3.1 in [159], which is in the spirit of Lemma A.3 below, that the function $u_R - \hat{u}$ is either strictly positive, strictly negative, or identically equal to zero in $B_R$, and obviously the latter case occurs. For completeness purposes, as well as for future reference, we will draw the same conclusion by an alternative and, to our opinion, more elementary approach: The function

$$v \equiv u_R - \hat{u}$$

solves the linear equation

$$\Delta v + Q(x) v = 0, \quad x \in B_R,$$

where

$$Q(x) = \begin{cases} \frac{W'(\hat{u}(x)) - W'(u_R(x))}{u_R(x) - \hat{u}(x)}, & \text{if } \hat{u}(x) \neq u_R(x), \\ \\ -W''(u_R(x)), & \text{if } \hat{u}(x) = u_R(x). \end{cases} \tag{2.26}$$

On the other hand, since

$$v(x) = 0, \quad x \in B_R \backslash B_{(R-D)},$$

and $Q \in L^\infty(B_R)$, the unique continuation principle (see for instance [155]) yields that

$$v(x) = 0, \ x \in B_R.$$

(In this simple case of radial symmetry, we can also make use of the uniqueness theorem of ordinary differential equations to show that $v \equiv 0$). Therefore, estimate (2.3) holds. We remark that, if $W$ was *strictly* decreasing in $(\mu - d, \mu)$, then (2.3) follows at once from the general lemma in [14] (see also the second assertion of Lemma A.2 herein) and (2.24).

The approach that we just presented makes only partial use of the radial symmetry of the problem (in order to establish (2.24)), and may be applied to extend some results in [96] to the general case (without radial symmetry), see [234]. Moreover, it can be applied for the study of global minimizers of the analogous vector-valued energy functionals, as those appearing in [12], over $B_R$. In this case, it is known that global minimizers are radial, see [185], but monotonicity properties do not hold in general.

**Remark 2.4.** In the one dimensional case, i.e., when $n = 1$, the assertion of Remark 2.3 can be shown *without* assuming (2.25). As in the latter remark, we do not assume the monotonicity property (2.9) of $u_R$, just that it is even, and we will start from (2.24) which clearly implies that

$$u_R(R-D) \to \mu \quad \text{as} \quad R \to \infty. \tag{2.27}$$

Since the energy of $u_R$ is not larger than that of the even function given by

$$\breve{u}_R(x) = \begin{cases} u_R(x), & x \in [R-D, R], \\ \\ \frac{u_R(R-D)-\mu}{D}(x-R+D) + u_R(R-D), & x \in [R-2D, R-D], \\ \\ \mu, & x \in [0, R-2D], \end{cases} \tag{2.28}$$

it follows readily from **(a')** and (2.27) that

$$\int_{-R+D}^{R-D} \left\{ (u_R')^2 + W(u) \right\} dx \to 0 \ \text{ as } R \to \infty. \tag{2.29}$$

Hence, by **(a')** and the *clearing–out* Lemma 1 in [50] (noting that it continues to apply in our possibly degenerate setting), we have that

$$u_R \to \mu, \ \text{uniformly in } [-R+D, R-D], \ \text{as } R \to \infty. \tag{2.30}$$

The intuition behind the latter lemma, as applied in the case at hand, is that if the energy is sufficiently small in some place, then there are no spikes located there. Note that from (2.2), (2.6), in arbitrary dimensions, via standard interior elliptic regularity estimates [142] (see also Lemma A.1 in [49]), applied on balls of radius $\frac{D}{4}$ covering $B_{(R-D)}$, we have that $|\nabla u_R|$ remains uniformly bounded in $B_{(R-D)}$ as $R \to \infty$ (or see the gradient bound in (2.56) below). Thus, relation (2.30) can also be derived from **(a')** and (2.29) similarly to Theorem III.3 in [51], see also Lemma 3.2 in [248] (the point is that the "bad" intervals, where $u_R$ is away from $\mu$ must have size of order one (by the uniform gradient estimate), as $R \to \infty$, which is not possible by **(a')** and (2.29)). In contrast to the one dimensional case, in $n \geq 2$ dimensions, by the analog of (2.29), i.e.,

$$R^{1-n} J(u_R; B_{(R-D)}) \to 0 \ \text{ as } \ R \to \infty, \tag{2.31}$$

arguing again as in Theorem III.3 in [51], we can show the weaker property:

$$\text{Given } \alpha \in (0,1) \ \Rightarrow \ u_R \to \mu, \ \text{uniformly in } \bar{B}_{(R-D)} \backslash B_{\alpha R}, \ \text{as } R \to \infty. \tag{2.32}$$

We note that if $W''(\mu) > 0$, then (2.30) follows directly from (2.29), via (2.44) below and the Sobolev embedding

$$\|\mu - u_R\|_{L^\infty(-R+D, R-D)} \leq C \|\mu - u_R\|_{W^{1,2}(-R+D, R-D)},$$

with constant $C$ independent of $R \geq 2D$ (see Corollary 5.16 in [1]). One might be curious whether this simple argument can be extended to $n \geq 2$ dimensions. In this direction, we would like to mention that by using the pointwise estimate

$$(\mu - u_R(r))^2 \leq C_n r^{1-n} \|\mu - u_R\|^2_{W^{1,2}\left(B_{(R-D)}\right)} + \left(\frac{R-D}{r}\right)^{n-1} (\mu - u_R(R-D))^2, \ \ r \in (0, R-D),$$

which can be proven similarly as the classical Strauss radial lemma (see [237]), relation (2.27), and (2.31), we arrive again at (2.32). On the other side, as in [11], fixing $K$ and letting $R \to \infty$, we see from the monotonicity formula (2.58) below that $u_R \to \mu$, uniformly on $\bar{B}_K$, as $R \to \infty$ (see also Remark 2.12 below, and the compactness argument that follows). Suppose that for a sequence $R \to \infty$, there exist $r_R \in [0, R-D]$ such that $u_R(r_R) = \mu - 2\epsilon$. From (2.32) and our previous comment, we get that $R - r_R \to \infty$ and $r_R \to \infty$, as $R \to \infty$, respectively. As in the proof of Theorems 1.3 and 1.4 in [96], we let $v_R(s) = u_R(r_R + s)$, $s \in (-r_R, R - r_R)$, note that $v_R(0) = \mu - 2\epsilon$. Using (2.2), (2.6), together with standard elliptic regularity estimates and Sobolev embeddings (see [142]), passing to a subsequence, we find that $v_R \to V$ in $C^1_{loc}(\mathbb{R})$, where

$$V'' = W'(V), \quad 0 \leq V \leq \mu, \quad s \in \mathbb{R}, \quad V(0) = \mu - 2\epsilon. \tag{2.33}$$

Moreover, the solution $V$ is a minimizer of the energy

$$I(v) = \int_{-\infty}^{\infty} \left[\frac{1}{2}(v')^2 + W(v)\right] ds,$$

in the sense that $I(V+\phi) \leq I(V)$ for every $\phi \in C_0^\infty(\mathbb{R})$, see page 104 in [96]. Arguing as in the proof of De Giorgi's conjecture in low dimensions (see [22], [39], [97], [121], [137], [203]), we can prove that either $V$ is a constant with $W'(V) = 0$, $W''(V) \geq 0$ or $V'$ is nontrivial and has fixed sign. Since we are assuming that $W''(\mu) > 0$, the first scenario is ruled out at once from the last condition in (2.33); in the second scenario, it follows from phase analysis (see [27], [247]) that $V$ has to connect two equal wells of the potential $W$ at respective infinities, one of them being $\mu$, but this is impossible since $W(t) > 0,\ t \in [0, \mu)$. Consequently, if we assume that $W''(\mu) > 0$, assertion (2.3) can be deduced in this manner from (2.24) *without* making use of (2.9) for *all* $n \geq 1$.

**Remark 2.5.** If $W \in C^{2,\alpha}(\mathbb{R})$, $0 < \alpha < 1$, satisfies **(a')**,

$$W'(\rho_1) = 0, \quad W'(t) < 0,\ t \in (\rho_1, \mu),\ \text{for some } \rho_1 \in (0, \mu),$$

and (1.15), then Theorem 2 in [238] tells us that there exists a $\delta_1 \in (0, \mu)$ such that (2.6) has at most one solution such that

$$\max_{x \in \bar{B}_R} u(x) \in (\mu - \delta_1, \mu) \text{ and } -\mu < u(x) < \mu,\ x \in B_R,$$

for all $R > 0$. Therefore, under these assumptions on $W$, in view of (2.2) and (2.3) which hold for all global minimizers (with the same $R'$), we conclude that there exists a unique global minimizer of (2.1), if $R$ is sufficiently large.

On the other side, if in addition to **(a')**, the stronger assumption $W''(\mu) > 0$ holds (in other words **(c)**), then a simple proof of the uniqueness of the global minimizer, satisfying (2.2), for large $R$, can be given as follows: One first shows that if a solution of (2.6) satisfies (2.2), (2.3), and (2.19) (recall (2.18)), then it is *asymptotically* stable for large $R > 0$ (we will give a short self–contained proof of this in the sequel). Then, suppose that $u_1$ and $u_2$ are two distinct global minimizers of (2.1), satisfying (2.2). By the proof of Lemma 2.1, they satisfy (2.2), (2.3), and (2.19), uniformly (independent of the choice of minimizers) as $R \to \infty$. Thanks to Lemma 3.1 in [159] (see also Lemma A.3 herein), without loss of generality, we may assume that $u_1(x) < u_2(x)$, $x \in B_R$ (in the problem at hand, we can also assume this when dealing with stable solutions). On the other hand, by the mountain pass theorem or the theory of monotone dynamical systems (see [100], [188] respectively, and Section 7 herein), we infer that there exists an *unstable* solution $\hat{u}_1$ of (2.6) such that $u_1(x) < \hat{u}_1(x) < u_2(x),\ x \in B_R$. In particular, the unstable solution enjoys the asymptotic behavior of global minimizers, as $R \to \infty$, and thus is asymptotically stable (by our previous discussion); a contradiction. A related uniqueness proof, based on a dynamical systems argument (but not of monotone nature), can be found in [8].

Here, for completeness, assuming that $W''(\mu) > 0$, we will show that solutions $u_R$ of (2.6) which satisfy (2.2), (2.3), and (2.19) are *asymptotically* stable if $R$ is sufficiently large. Our argument is inspired from [26] where, in particular, under the additional assumption **(b)** with strict inequality, it was applied to (7.9) below on a smooth bounded domain with large $\lambda$. We will ague by contradiction. Suppose that, for a sequence $R \to \infty$, the principal eigenvalue $\mu_R$ of the linearized operator about $u_R$ is non-positive, i.e,

$$\mu_R \leq 0. \tag{2.34}$$

It is well known that $\mu_R$ is simple and that the corresponding eigenfunction $\varphi_R$ (modulo normalization) may be chosen to be positive in $B_R$, see for instance Theorem 8.38 in [142]. We have

$$-\Delta\varphi_R + W''(u_R)\varphi_R = \mu_R\varphi_R \text{ in } B_R;\ \varphi_R = 0 \text{ on } \partial B_R, \tag{2.35}$$

and we normalize $\varphi_R$ by imposing that

$$\|\varphi_R\|_{L^\infty(B_R)} = 1. \tag{2.36}$$

We note that $\varphi_R$ is radially symmetric (and so is every eigenfunction that is associated to a non-positive eigenvalue, see [151], [181], because **(a')** and Hopf's boundary point lemma yield that $u'_R(R) < 0$). For future reference, observe that testing (2.35) by $\varphi_R$ yields the uniform (in $R$) lower bound:

$$\mu_R \geq -\max_{t\in[0,\mu]} |W''(t)|. \tag{2.37}$$

Now, by virtue of (2.3) and the positivity of $W''(\mu)$, there exists a constant $D > 0$ such that

$$W''(u_R) \geq \frac{W''(\mu)}{2} > 0 \text{ on } \bar{B}_{(R-D)},$$

for large $R > 0$. So, from (2.34), (2.35), and (2.36), we obtain that there exist $z_R \in [R-D, R]$ such that $\varphi_R(z_R) = 1$, $\varphi'_R(z_R) = 0$, and $\varphi''_R(z_R) \leq 0$, for large $R$ (along the sequence). As in the proof of Lemma 2.1, making use of (2.19), (2.34), (2.35), (2.36), and (2.37), passing to a subsequence, we get that $\varphi_{R_i}(R_i - \cdot) \to \Phi(\cdot)$ in $C^1_{loc}[0,\infty)$, $\mu_{R_i} \to \mu_* \leq 0$, and $R_i - z_{R_i} \to \mathbf{z} \in [0, D]$, as $i \to \infty$, such that

$$-\Phi'' + W''(\mathbf{U}(r))\Phi = \mu_*\Phi,\ r \in (0,\infty);\ \Phi(0) = 0,\ \Phi(\mathbf{z}) = \|\Phi\|_{L^\infty(0,\infty)} = 1, \tag{2.38}$$

where $\mathbf{U}$ is as in (1.12). On the other hand, differentiating (1.12), multiplying the resulting identity by $\frac{\Phi^2}{\mathbf{U}'}$ (recall (1.19)) and integrating by parts over $(0,\infty)$, we arrive at $\mu_* \geq 0$ (see also Proposition 3.1 in [203]); to be more precise, one first multiplies by $\frac{\zeta_m^2}{\mathbf{U}'}$, with $\zeta_m \in C_0^\infty(0,\infty)$ such that $\zeta_m \to \Phi$ in $W_0^{1,2}(0,\infty)$, and then lets $m \to \infty$. A different way to see that $\mu_* \geq 0$ is to note that the linear operator defined by the lefthand side of (2.38) is an unbounded, self-adjoint operator in $L^2(0,\infty)$ with domain $W_0^{1,2}(0,\infty)\cap W^{2,2}(0,\infty)$, having as continuous spectrum the interval $[W''(\mu),\infty)$ and principal eigenvalue zero (by the positivity of $\mathbf{U}'$), see also Remark 2.8 in [10] or Proposition 1 in [150] or [220]. In other words, recalling (2.34), we have

$$-\Phi'' + W''(\mathbf{U}(r))\Phi = 0,\ \Phi > 0,\ r \in (0,\infty);\ \Phi(0) = 0,\ \Phi(\mathbf{z}) = \|\Phi\|_{L^\infty(0,\infty)} = 1.$$

The above linear second order equation has the following two independent solutions:

$$\mathbf{U}'(r) \text{ and } \mathbf{U}'(r)\int_0^r \frac{1}{[\mathbf{U}'(s)]^2}ds,$$

see for example Lemma 3.2 in [32]. It is easy to see that the second solution grows unbounded as $r \to \infty$ (plainly apply l'hospital's rule), and thus $\Phi$ has to be $\|\mathbf{U}'\|^{-1}_{L^\infty(0,\infty)}\mathbf{U}'$. Since $\Phi(0) = 0$, whereas $\mathbf{U}'(0) = \sqrt{2W(0)} > 0$, we have reached a contradiction.

For further information on "asymptotic" uniqueness of positive solutions, in arbitrary domains, we refer to Remark 7.1 below.

In the case where uniqueness of a stable solution, satisfying (2.2), holds for $R > R_0 \geq 0$ (recall Remark 1.3 and see Remark 7.1 below), it is easy to see that the family $\{u_R\}_{R>R_0}$ is nondecreasing with respect to $R$, namely

$$u_{R_2}(x) > u_{R_1}(x), \quad x \in \bar{B}_{R_1}, \quad \forall\, R_2 > R_1 > R_0, \tag{2.39}$$

see Lemma 1 in [99]. Moreover, as in Lemma 2 in [99], we have that

$$u_R(R-r) \leq \mathbf{U}(r), \quad r \in [0,R], \ \forall R > R_0, \tag{2.40}$$

(plainly observe that, thanks to (1.12) and (1.19), the function $\mathbf{U}(R-r)$ is a weak upper solution to (2.6) in the sense of [35]). We note that, arguing as in Remark 2.3 (see also Lemma A.3 in Appendix A), it follows that (2.39) holds *without* assuming uniqueness (plainly observe that $J\left(\max\{u_{R_2}, u_{R_1}\}; B_{R_2}\right) \leq J(u_{R_2}, B_{R_2})$, see also Lemma 5.3 in [138]). Moreover, similarly to Proposition 1.5 in [69], based on the gradient estimate (2.56), one can show (2.40) without assuming uniqueness.

**Remark 2.6.** If in addition to **(a')**, we assume that $W'(0) = 0$, $W''(0) < 0$, and $W \in C^3$ ($W'''$ bounded for $t > 0$ small is enough), then (2.6) admits a nontrivial positive solution, which is a global minimizer of $J(\cdot; B_R)$ in $W_0^{1,2}(B_R)$, as long as $R > R_c$, where

$$R_c = \sqrt{-\frac{\lambda_1}{W''(0)}}, \tag{2.41}$$

and $\lambda_1$ denotes the principal eigenvalue of $-\Delta$ in $W_0^{1,2}(B_1)$ (an analogous result holds for (7.9) below). To see this, let $\varphi_1$ denote the associated eigenfunction with the normalization $\varphi_1(0) = 1$ ($\varphi_1$ is radially decreasing). Then, the pair

$$\lambda_R = \lambda_1 R^{-2} \quad \text{and} \quad \varphi_R(x) = \varphi_1(R^{-1}x) \tag{2.42}$$

is the principal eigenvalue and eigenfunction of $-\Delta$ in $W_0^{1,2}(B_R)$ such that $\varphi_R(0) = 1$. Now, the desired conclusion follows at once by noting that

$$J\left(\varepsilon\varphi_R; B_R\right) = J\left(0; B_R\right) + \frac{\varepsilon^2}{2}\int_{B_R} \varphi_R^2\left(\lambda_1 R^{-2} + W''(0) + \mathcal{O}(\varepsilon)\right) dx \quad \text{as } \varepsilon \to 0^+,$$

which implies that zero is not a global minimizer if $R > R_c$ (see also Example 5.11 in [24], Theorem 2.19 in [29], Lemma 2.1 in [104] and Proposition 1.3.3 in [113]; note also that $\varepsilon\varphi_R$, with $R > R_c$, is a lower solution to (2.6) for small $\varepsilon > 0$). (If $W$ is even, one can construct a plethora of sign-changing solutions, for large $R$, not necessarily radial, by noting that $J(u; B_R) < J(0; B_R)$ for $u \in \mathrm{Span}\left\{\varphi_1(R^{-1}x), \cdots, \varphi_k(R^{-1}x)\right\}$, $k \geq 1$, and $\|u\|_{L^2(B_R)}$ sufficiently small, where $\varphi_i$ denote eigenfunctions of the Laplacian in $W_0^{1,2}(B_1)$ (normalized so that $\|\varphi_i(R^{-1}x)\|_{L^2(B_R)} = 1$ and $\int_{B_1} \varphi_i\varphi_j dx = 0$ if $i \neq j$), corresponding to the first $k$ eigenvalues (counting multiplicities), and applying Theorem 8.10 in [211]; see also [7] and Theorem 10.22 in [24]).

If we further assume that

$$W'(t) \geq W''(0)t, \quad t \geq 0, \tag{2.43}$$

then (2.6), for $R \in (0, R_c)$, has no positive solution as can be seen by testing the equation by $\varphi_R$.

Under some different conditions, which are compatible with **(a')**, and are satisfied for example by the nonlinearity in (1.10), there exists an $R_c' > 0$ such that (2.6) has exactly one positive solution for $R = R_c'$ and exactly two for $R > R_c'$, the one is a global minimizer while the other is a mountain pass of the associated energy (see [201], [228], [250]).

**Remark 2.7.** By (2.10), via the coarea formula (see [115]), it follows that there exists a $\xi_R \in \left(\frac{R}{2}, R\right)$ such that

$$\int_{\partial B_{\xi_R}} \left\{\frac{1}{2}|\nabla u_R|^2 + W(u_R)\right\} dS \leq 2C_1 R^{n-2}, \quad R \geq C_1 C_2^{-1} + 2.$$

This observation makes no use of the radial symmetry of $u_R$, and is motivated from the proof of the corollary in [14]. In regard to the latter comment, it might be useful to recall our Remark 2.4 and compare with the arguments of [14].

**Remark 2.8.** In case a $C^2$ potential $W$ satisfies $W(0) = 0$ and the domain $\Omega$ has $C^1$ boundary, is bounded, and star-shaped with respect to some point in its interior, the well known Pohozaev identity easily implies that there does not exist a nontrivial solution of (1.2) such that $W(u(x)) \geq 0,\ x \in \Omega$ (see for instance relation (11) in [17], a reference which is in accordance with our notation). Actually, relation (11) in the latter reference holds true for the elliptic system that corresponds to (1.2) (with the obvious notation), and an analogous nonexistence result holds in that situation as well.

**Remark 2.9.** Under the stronger assumptions **(a)** (or more generally **(a')**), **(b)**, and **(c)**, considered in [134] (recall the introduction herein), motivated from the proof of Lemma 3 in [182] (see also [222] and the remarks following Lemma 2.1 in [77]), we can give a streamlined proof of relation (2.12) as follows: Note first that, thanks to **(a')** and **(c)**, there exists a positive constant $c_0$ such that

$$W(t) \geq c_0(\mu - t)^2, \quad 0 \leq t \leq \mu. \tag{2.44}$$

Then, bounds (2.2), (2.7), and the above relation yield that

$$\int_{B_R} (\mu - u_R)^2 dx \leq c_1 R^{n-1}, \quad R \geq 2, \tag{2.45}$$

where the positive constant $c_1$ depends only on $W$ and $n$. Next, note that assumption **(b)**, bound (2.2), and the equation in (2.6), imply that the function $\mu - u_R$ is subharmonic in $B_R$, and thus we have

$$\Delta(\mu - u_R)^2 \geq 0 \ \text{ in } \ B_R, \quad R \geq 2.$$

In other words, the function $(\mu - u_R)^2$ is also subharmonic in $B_R$. Consequently, by (2.45) and the mean value inequality of subharmonic functions (see Theorem 2.1 in [142]) together with a simple covering argument (see also the general Theorem 9.20 in [142] and Chapter 5 in [192]), we deduce that

$$\max_{\bar{B}_{\frac{R}{2}}} (\mu - u_R)^2 \leq c_2 R^{-n} \int_{B_R} (\mu - u_R)^2 dx \leq c_3 R^{-1}, \quad R \geq 2, \tag{2.46}$$

where the positive constants $c_2,\ c_3$ depend only on $W$ and $n$. The latter inequality clearly implies the validity of (2.12). In passing, we note that the spherical mean of $(\mu - u_R)^2$ appearing in the above inequality is nondecreasing with respect to $R$, because of the subharmonic property, see [210].

The above argument makes no use of the fact that $u_R$ is radially symmetric. Moreover, it works equally well if instead of (2.44) we had $W(t) \geq c(\mu - t)^p,\ t \in [0, \mu]$, for some constants $c > 0$ and $p > 2$. In Appendix D, we will adapt this approach in order to simplify some arguments from Section 6 of the recent paper [13], where the De Giorgi oscillation lemma for subharmonic functions was employed instead of the mean value inequality. The former lemma roughly says that if a positive subharmonic function is smaller than one in $B_1$ and is "far from one" in a set of non trivial measure, it cannot get too close to one in $B_{\frac{1}{2}}$ (see for example [78]). An intriguing application of the techniques in the current remark is given in the following Remark 2.10.

**Remark 2.10.** When seeking solutions of (1.22) in $\mathbb{R}^3$ which are invariant under screw motion and whose nodal set is a helicoïd, assuming that $W$ is even, by introducing cylindrical coordinates, one is led to study positive solutions of

$$\partial_r^2 U + \frac{1}{r}\partial_r U + \left(1 + \frac{\lambda^2}{\pi^2 r^2}\right)\partial_s^2 U - W'(U) = 0, \tag{2.47}$$

in the infinite half strip $\{(r, s) \in (0, \infty) \times (0, \lambda)\}$, vanishing on the boundary of $[0, \infty) \times [0, \lambda]$, where $\lambda$ corresponds to a dilation parameter of a fixed helicoïd. More specifically, such solutions $U$ give rise to solutions $u$ of (1.22) which vanish on the helicoïd that is parameterized by

$$\left\{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3\ :\ z = \frac{\lambda}{\pi}\theta\right\},$$

see [104] for the details. In the latter reference, assuming that $W''(0) < 0$ and (2.43), it was shown that there exists an explicit constant $\lambda^* > 0$ such that the above problem has a positive solution $U_\lambda$ if and only if $\lambda > \lambda^*$ ($\lambda^*$ is actually equal to $2R_c$, where $R_c$ is given from (2.41) with $n = 1$).

Here, motivated from our previous Remark 2.9, we will study this problem for large values of $\lambda$ under complementary conditions on $W$ (in particular, without assuming that $W''(0) < 0$). Due to the presence of singularities in the equation (2.47) at $r = 0$, as in Lemma 3.4 in [112], we will first consider the approximate (regularized) problem

$$\Delta_{\mathbb{R}^2} U + \left(1 + \frac{\lambda^2}{\pi^2 |x|^2}\right)\partial_s^2 U - W'(U) = 0, \tag{2.48}$$

in $\{\xi < |x|,\ s \in (0, \lambda)\}$ with zero conditions on $\{|x| = \xi,\ s \in [0, \lambda]\}$ and $\{\lambda = 0,\ |x| \geq \xi\}$, $\{\lambda = 1,\ |x| \geq \xi\}$, with $\xi$ small (this was skipped in [104]). Then, we consider equation (2.48) in the annular cylinder $\{\xi < |x| < R,\ s \in (0, \lambda)\}$, imposing that $U$ also vanishes on $|x| = R$. Assuming **(a')**, as in Lemma 2.1, by minimizing the energy

$$E(V) = \frac{1}{2}\int \left\{|\nabla_x V|^2 + \left(1 + \frac{\lambda^2}{\pi^2 |x|^2}\right)|\partial_s V|^2 + 2W(V)\right\} dx ds \tag{2.49}$$

in $W_0^{1,2}\left((B_R \backslash B_\xi) \times (0, \lambda)\right)$ (with the obvious notation), but this time *in the radially symmetric class* with respect to $|x|$ (minimizers in this class are critical points in the usual sense, see [204]), we find a solution $U_{\xi,R,\lambda}$ of (2.48), satisfying the prescribed Dirichlet boundary conditions, such that $0 \leq U_{\xi,R,\lambda}(|x|, s) \leq \mu$ on $\left(\bar{B}_R \backslash B_\xi\right) \times [0, \lambda]$ (see Lemma A.2 below). Moreover, as in the proof of Lemma 2.1, we have

$$E\left(U_{\xi,R,\lambda}\right) \leq CR\lambda, \quad R \geq 2,\ \lambda \geq 2,\ \xi \leq 1, \tag{2.50}$$

with $C$ independent of $\xi,\ R,\ \lambda$ (for this, it is convenient to use a separable test function of the form $\eta(r)\vartheta(s)$, see also [104]). Hence, again as in Lemma 2.1, we have that $0 < U_{\xi,R,\lambda}(|x|,s) < \mu$, if $\xi \leq |x| \leq R,\ s \in [0,\lambda]$, for all $\xi \leq 1,\ \lambda \geq 2$, provided that $R$ is sufficiently large (note that $E(0) = \lambda\pi(R^2-\xi^2)W(0)$). Using the standard compactness argument, letting $\xi \to 0$ and $R \to \infty$ (along a sequence), we are left with a solution $U_\lambda$ of (2.47) in the infinite half strip $(0,\infty)\times(0,\lambda)$, with zero conditions on its boundary, such that $0 \leq U_\lambda \leq \mu$ on the half strip. The latter relation leaves open the possibility of $U_\lambda$ being identically zero. However, $U_\lambda$ is a minimizer of the energy in (2.49), in the sense of (2.72) below (since it is the limit of a family of minimizers, see also page 104 in [96]). So, with the help of a suitable energy competitor (see for example (2.28) or (2.70)), for any two–dimensional ball $B_{\frac{\lambda}{3}}(q)$ of radius $\frac{\lambda}{3}$ that is contained in $(\lambda,\infty)\times(1,\lambda-1)$, we have

$$\int_{B_{\frac{\lambda}{3}}(q)} W(U_\lambda)drds \leq C\lambda^2,$$

with constant $C>0$ independent of large $\lambda$. If we further assume that conditions **(b)** and **(c)** hold, noting that

$$\partial_r^2(\mu-U_\lambda)+\frac{1}{r}\partial_r(\mu-U_\lambda)+\left(1+\frac{\lambda^2}{\pi^2r^2}\right)\partial_s^2(\mu-U_\lambda)\geq 0,$$

and that the coefficients of the elliptic operator above satisfy

$$\frac{1}{r}\leq\lambda^{-1},\quad 1\leq 1+\frac{\lambda^2}{\pi^2r^2}\leq 1+\pi^{-2}\ \text{ on } B_{\frac{\lambda}{3}}(q),$$

the arguments in Remark 2.9 can be applied to show that

$$U_\lambda\to\mu,\ \text{uniformly on } \bar{B}_{\frac{\lambda}{6}}(q),\ \text{ as } \lambda\to\infty.$$

Since $q$ was any point with coordinates $r>\frac{4\lambda}{3}$ and $s\in\left(\frac{\lambda}{3}+1,\frac{2\lambda}{3}-1\right)$, we deduce that

$$U_\lambda\to\mu,\ \text{uniformly on } [2\lambda,\infty)\times\left[\frac{\lambda}{6}+1,\frac{5\lambda}{6}-1\right],\ \text{ as } \lambda\to\infty.$$

Studying the existence and asymptotic behavior of $U_\lambda$, as $\lambda\to\infty$, assuming *only* **(a')**, is left as an interesting open problem.

For future reference, let us prove here the following lemma.

**Lemma 2.2.** Suppose that $W\in C^2$ satisfies **(a')**, (1.8), and (1.9). Let $u_R$ denote a family of solutions to (2.6), not necessarily global minimizers, such that (2.2) holds. Then, we have that $u_R$ are radial as well as the validity of relations (2.3), (2.9), (2.13), and (2.19), *uniformly* with respect to the family $u_R$.

*Proof.* Since $u_R$ is positive, thanks to [140], we have that $u_R$ is radial and that (2.9) holds. Similarly to Lemma 2.1, the functions $U_R$, defined through (2.18), satisfy (2.21) for some $V$ which solves (2.22) with $0\leq V(s)\leq\mu,\ s\geq 0$.

We claim that $V$ is nontrivial. In the case where $W'(0)<0$, this is clear. If $W'(0)=0$ and $W''(0)<0$ (keep in mind (1.8)), we argue as follows. Let $\lambda_R,\ \varphi_R$ be as in (2.42). In view of (1.9), we have

$$W'(t)\leq -ct,\quad t\in\left[0,\frac{\mu}{2}\right],$$

for some $c > 0$ (necessarily $c \leq -W''(0)$). Observe that the functions $\tau\varphi_R$, with $\tau \in \left[0, \frac{\mu}{2}\right]$, satisfy

$$-\Delta(\tau\varphi_R) + W'(\tau\varphi_R) \leq \frac{\lambda_1}{R^2}\tau\varphi_R - c\tau\varphi_R \leq 0 \ \text{ in } B_R,$$

if $R > \sqrt{\frac{\lambda_1}{c}}$ (we also used that $\varphi_R(x) \leq \varphi_R(0) = 1$, $x \in B_R$). Consider a ball $B_R$ with $R > 3\sqrt{\frac{\lambda_1}{c}}$ and another ball $B_{\sqrt{\frac{4\lambda_1}{c}}}(p) \subset B_R$ such that they touch at one point on $\partial B_R$. By Serrin's sweeping technique (see the references in the first proof of Theorem 1.2 below), keeping in mind that $u_R'(R) < 0$ (by Hopf's lemma), it follows that

$$u_R(x) \geq \frac{\mu}{2}\varphi_{\sqrt{\frac{4\lambda_1}{c}}}(x-p), \quad x \in B_{\sqrt{\frac{4\lambda_1}{c}}}(p).$$

(In fact, since $u_R$ is radially symmetric, the above bound holds for all $p \in B_R$ such that $\text{dist}(p, \partial B_R) = 2\sqrt{\frac{\lambda_1}{c}}$). This lower bound certainly ensures that $V$ is nontrivial. Then, by the strong maximum principle, we deduce that

$$0 < V(s) < \mu, \quad s > 0. \tag{2.51}$$

On the other hand, from **(a')**, (1.9), and the phase portrait of the ordinary differential equation (see for example [27]), the only solution of (2.22) which satisfies (2.51) is $\mathbf{U}$, as described in (1.12). By the uniqueness of the limiting function, we infer that (2.21) holds for $R \to \infty$. So, we have proven that (2.19) and in turn (2.3), (2.13) hold for each such family of solutions.

The fact that they hold *uniformly* with respect to the family $\{u_R\}$ follows plainly from the observation that every such family is uniformly bounded in $C^2(\bar{B}_R)$ with respect to $R$. The latter property follows from the fact that $0 < u_R < \mu$ in $B_R$ and a standard bootstrap argument involving elliptic regularity (the gradient bounds for $u_R$ follow from elliptic estimates [142] applied on balls of radius one covering $B_R$).

The proof of the lemma is complete. □

An extension of Lemma 2.1 can be shown, allowing the possibility $W'(0) \geq 0$, provided that the potential $W$ satisfies:

**(a"):** There exist constants $\mu_- \leq 0$ and $\mu > 0$ such that

$$0 = W(\mu) < W(t), \ t \in [\mu_-, \mu), \quad W(t) \geq 0, \ t \in \mathbb{R},$$

$$W(2\mu_- - t) \geq W(t), \ t \in [\mu_-, \mu] \text{ or } W'(t) < 0, \ t < \mu_-.$$

Note that **(a")** reduces to **(a')** when $\mu_- = 0$. We point out that the existence of $\mathbf{U}$, as in (1.12), also holds under **(a")**.

Below, we state such a result which seems to be new and of independent interest. In particular, it will be useful in Sections 6 and 10.

**Lemma 2.3.** Assume that $W \in C^2$ satisfies condition **(a")**. Let $\epsilon \in (0, \mu)$ and $D > D'$, where $D'$ is as in (1.11). Then, there exists a positive constant $R' > D$, depending only on $\epsilon$, $D$, $W$, and $n$, such that there exists a global minimizer $u_R$ of the energy functional in (2.1) which satisfies (2.2), (2.3), and (2.4), provided that $R \geq R'$. (As before, we assume that $W$ has been appropriately extended outside of a large compact interval). (We have chosen to keep some of the notation from Lemma 2.1).

*Proof.* The existence of a minimizer $u_R$, which solves (2.6), and satisfies

$$\mu_- < u_R(x) < \mu, \ x \in B_R,$$

follows as in the proof of Lemma 2.1. The main difference with the proof of Lemma 2.1 is that the above relation does not exclude the possibility of the minimizer $u_R$ taking non-positive values. In particular, the method of moving planes (see [61], [92], [140]) is not applicable in order to show that $u_R$ is radially symmetric and decreasing. (Nevertheless, it is known that nonnegative solutions of (2.6), with $n \geq 2$, are actually positive in $B_R$ and so the method of moving planes is still applicable in that situation, see [207] and the references therein). Not all is lost however. As we have already remarked in the proof of Lemma 2.1, if $n \geq 2$, the stability of $u_R$ (as a global minimizer) implies that it is radially symmetric, see Lemma 1.1 in [9], Remark 3.3 in [67], Proposition 2.6 in [96]; for an elegant proof that exploits the fact that $u_R$ is a global minimizer, see Corollary II.10 in [185] (see also [149] and Appendix C in [252]). In [89], see also Proposition 10.4.1 in [81] and Proposition 3.4 in [194], it has additionally been shown that stable solutions have constant sign, and hence are radially monotone by the method of moving planes. For the reader's convenience, we will show that $u_R(r)$ is a decreasing function of $r$, namely that (2.9) holds true, by a far more elementary argument. In view of (2.11), which still holds for the case at hand (by virtue of radial symmetry alone), it suffices to show that $u_R'(r) \neq 0, \ r \in (0, R]$. We will follow the part of the proof of Lemma 2 in [68] which dealt with problem (1.22) with $n \geq 3$ (see also Proposition 1.3.4 in [113]), and in fact show that it continues to apply for $n \leq 2$. To this end, we have not been able to adapt the approach of Lemma 1 in [8], which basically consists in multiplying (2.54) below by $V^+ \equiv \max\{V, 0\} \in W^{1,2}(B_R)$ and integrating the resulting identity by parts over $B_R$, since in the problem at hand $V(R) = u_R'(R)$ may be positive. Let

$$V \equiv u_R',$$

and suppose, to the contrary, that $V(R_0) = 0$ for some $R_0 \in (0, R]$. We will show that the function

$$\tilde{V}(r) = \begin{cases} V(r), & r \in [0, R_0], \\ 0 & r \in [R_0, R], \end{cases} \tag{2.52}$$

belonging in $W_0^{1,2}(B_R)$, satisfies

$$\int_{B_R} \left\{ |\nabla \tilde{V}|^2 + W''(u_R)\tilde{V}^2 \right\} dx < 0, \tag{2.53}$$

which clearly contradicts the stability of $u_R$. Differentiating (2.6) with respect to $r$, we arrive at

$$-\Delta V + W''(u_R)V + \frac{n-1}{r^2}V = 0, \quad x \in B_R \backslash \{0\}. \tag{2.54}$$

Let $\zeta$ be a smooth function such that

$$\zeta(t) = \begin{cases} 0, & t \in [0, 1], \\ 1, & t \in [2, \infty). \end{cases}$$

Multiplying (2.54) by $\zeta\left(\frac{r}{\varepsilon}\right)V(r)$, with $\varepsilon > 0$ small, and integrating the resulting identity by parts over $B_{R_0}$ (recall that $V(R_0) = 0$), we find that

$$\int_{B_{R_0}} \left\{\zeta\left(\frac{r}{\varepsilon}\right)|\nabla V|^2 + \frac{1}{\varepsilon}V\zeta'\left(\frac{r}{\varepsilon}\right)\left(\frac{x}{r}\cdot\nabla V\right) + \zeta\left(\frac{r}{\varepsilon}\right)W''(u_R)V^2 + \zeta\left(\frac{r}{\varepsilon}\right)\frac{n-1}{r^2}V^2\right\}dx = 0. \tag{2.55}$$

Note that

$$\left|\int_{B_{R_0}} \frac{1}{\varepsilon}V\zeta'\left(\frac{r}{\varepsilon}\right)\left(\frac{x}{r}\cdot\nabla V\right)dx\right| \leq C\varepsilon^{-1}\int_{\varepsilon}^{2\varepsilon} r^{n-1}dr \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

since the constant $C > 0$ does not depend on $\varepsilon$. (Note that we have silently assumed that $N \geq 2$, since in the case $N = 1$ we can plainly multiply (2.54) by $V$ and then integrate by parts over $(-R_0, R_0)$). So, letting $\varepsilon \to 0$ in (2.55), and employing Lebesgue's dominated convergence theorem (see for instance page 20 in [115]), it readily follows that

$$\int_{B_{R_0}} \left\{|\nabla V|^2 + W''(u_R)V^2 + \frac{n-1}{r^2}V^2\right\}dx = 0,$$

where in order to obtain the last term we used that $|V(r)| \leq C'r$, $r \in [0, R]$, with constant $C' > 0$ depending only on $R$ (keep in mind that $u_R \in C^2[0, R]$ with $u_R''(0) = \frac{1}{n}W'(u_R(0))$, see for instance page 72 in [247]). From the above relation, via (2.52), we get (2.53). We have thus arrived at the desired contradiction. Consequently, the monotonicity relation (2.9) also holds for the more general case at hand. The rest of the argument follows word by word the proof of Lemma 2.1, and is therefore omitted.

The proof of the lemma is complete. □

**Remark 2.11.** Suppose that $u_R$ is as in Lemma 2.1 or Lemma 2.3, and $E_R$ as defined in (2.14). From (2.15), it follows that

$$E_R(r) < E_R(0) = -W(u_R(0)) < 0, \quad r \in [0, R],$$

i.e.,

$$\frac{1}{2}[u_R'(r)]^2 < W(u_R), \quad r \in (0, R], \tag{2.56}$$

recall **(a')** and that $u_R'(0) = 0$, see also Remark 4 in [8] for a related discussion. In passing, we note that every bounded solution of (1.22) satisfies

$$\frac{1}{2}|\nabla u|^2 \leq W(u), \quad x \in \mathbb{R}^n, \tag{2.57}$$

provided that $W$ is nonnegative.The proof of this gradient bound, originally due to L. Modica [191], is much more complicated than that of its radially symmetric counterpart (2.56). We refer the interested reader to [74] and Lemma 4.1 in [83] (see also Proposition 11.1 herein). In turn, making use of the gradient bound (2.56), we can establish the monotonicity formula

$$\frac{d}{dr}\left(\frac{1}{r^{n-1}}\int_{B_r}\left\{\frac{1}{2}|\nabla u_R|^2 + W(u_R)\right\}dx\right) > 0, \quad r \in (0, R), \tag{2.58}$$

see [11] for a modern approach as well as the older references therein which include [74]. In passing, we note that a similar monotonicity formula holds true for solutions of (1.22), and a weaker one (with the exponent $n-1$ replaced by $n-2$) holds in the case of the corresponding systems, see again [11] and the references therein or [77]. Now, making use of (2.7) and the above relation, we find that

$$\frac{1}{K^{n-1}}\int_{B_K}\left\{\frac{1}{2}|\nabla u_R|^2+W(u_R)\right\}dx<C_1 \quad \forall\ K\in(0,R), \quad R\geq 2.$$

We have therefore provided a proof (of a sharper version) of (2.8). It also follows from (2.58) that $R^{1-n}J(u_R;B_R)$ remains bounded from below by some positive constant, as $R\to\infty$ (compare with (2.7)). If $W''(\mu)>0$, making use of (2.19), it is not hard to determine a constant to which $R^{1-n}J(u_R;B_R)$ converges as $R\to\infty$ (see [25] and [138]), recall also the last part of Remark 2.1 (in order to avoid confusion, we point out that we have *not* shown that the latter function is increasing in $R$). In this regard, we also refer to Theorem 7.10 in [55] where functionals of the form (2.1) are shown to converge (in an appropriate variational sense) to functionals involving the perimeter of the domain.

**Remark 2.12.** Here, for completeness, we sketch an argument related to the proof of Lemma 2.3. By (2.2), elliptic estimates (see [142]), and a standard compactness argument, it follows readily that $u_R$ converges, up to a subsequence $R_i\to\infty$, uniformly on compact subsets of $\mathbb{R}^n$ to a radially symmetric solution $U$ of (1.22) such that $0\leq U(x)\leq\mu,\ x\in\mathbb{R}^n$. Moreover, arguing as in page 104 of [96], this solution is a global minimizer of (1.22) in the sense of (2.72) below, with $\Omega=\mathbb{R}^n$, see also [69], [159].

On the other hand, it is known that (1.22), for *any* $W\in C^2$, does not have nonconstant bounded, radial global minimizers (see [246]). This property is also related to the nonexistence of nonconstant "bubble" solutions to (1.22) with $W\geq 0$ vanishing nondegenerately at a finite number of points, namely solutions that tend to one of these points as $|x|\to\infty$, see Theorem 2 in [79], Chapter 4 in [230] and recall Remark 2.8 (keep in mind that stable solutions of (1.22) are radially monotone and tend to a local or global minimum of $W$, as $r\to\infty$, see [68]). In passing, we note that if $n\leq 10$ then nonconstant radial solutions of (1.22), with $W\in C^2$ arbitrary, are unstable (see [68]). Under certain assumptions on $W$, satisfied by the Allen-Cahn potential (1.23) for example, it was shown in [145] (see also [52]) that nonconstant radial solutions of (1.22) tend in an oscillatory manner to zero as $r\to\infty$ and thus are unstable. More generally, the nonexistence of nonconstant finite energy solutions to (1.22) with $W\geq 0$ holds, see [11] or [150] where this property is refereed to as a theorem of Derrick and Strauss. Related nonexistence results for nonnegative solutions can be found in Sections 5 and 6 herein.

Obviously $U\equiv\mu$ (recall **(a')**) and, by the uniqueness of the limit, the convergence holds for *all* $R\to\infty$. We conclude that, given any $K>1$, we have $u_R\to\mu$, uniformly in $B_K$, as $R\to\infty$. The main advantage of this approach is that it continues to work when (2.6) is replaced by $\Delta u=F_R(|x|,u)$, with a suitable $F_R(|x|,u)$ which converges uniformly over compact sets of $[0,\infty)\times\mathbb{R}$ to a $C^1$ function $F(u)$ (the point being that $\frac{d}{dr}F_R(r,u)$ may be negative somewhere, and (2.9) may fail in $B_R$).

The following lemma is motivated from Lemma 3.3 in [38].

**Lemma 2.4.** Assume that $W$ satisfies conditions **(a")** and (1.15). Let $\epsilon\in(0,\mu)$ be any number such that

$$W''(t)\geq 0 \ \text{ on } \ [\mu-\epsilon,\mu]. \tag{2.59}$$

Then, the global minimizers $u_R$ that are provided by Lemmas 2.1 and 2.3 satisfy

$$-W'\left(u_R(0)\right)\leq\tilde{C}R^{-2}, \tag{2.60}$$

where the constant $\tilde{C} > 0$ depends only on $n$, provided that $R \geq R'$, where $R'$ is as in the latter lemmas.

*Proof.* Let $D$ be as in the assertions of Lemmas 2.1 and 2.3. Thanks to (2.2), (2.3), (2.6), (2.9), and (2.59), we have

$$\Delta u_R = W'(u_R) \leq W'(u_R(0)) \quad \text{on} \quad \bar{B}_{(R-D)},$$

if $R \geq R'$.

For such $R$, let $Z_R$ be the solution of

$$\Delta Z_R = W'(u_R(0)) \quad \text{in } B_{(R-D)}; \quad Z_R = 0 \quad \text{on } \partial B_{(R-D)}.$$

By scaling, one finds that

$$\max_{|x| \leq R-D} Z_R(x) = Z_R(0) = -z(0) W'(u_R(0))(R-D)^2,$$

for $R \geq R'$, where $z$ is the solution of

$$\Delta z = -1 \quad \text{in } B_1; \quad z = 0 \quad \text{on } \partial B_1.$$

By the maximum principle, we deduce that

$$Z_R(x) \leq u_R(x) < \mu, \quad x \in B_{(R-D)}, \quad R \geq R'.$$

In particular, by setting $x = 0$ in the above relation, we get (2.60).

The proof of the lemma is complete. □

**Remark 2.13.** In the special case where $W$ satisfies (1.9), $W'(0) = 0$, $W''(0) < 0$, $W'(\mu) = 0$, and $W''(\mu) > 0$, the estimate of Lemma 2.4 becomes that of Lemma 3.1 in [180].

Under conditions **(a”)** and (1.15), the global minimizers that are provided by Lemmas 2.1 and 2.3 are *asymptotically* stable, if $R$ is sufficiently large. This property is a direct consequence of the following proposition, which will play an essential role in our proofs of Theorems 6.1-6.2 and Propositions 6.2-6.3 below, as well as in our first proof of Theorem 1.2.

**Proposition 2.1.** Assume that **(a”)** and (1.15) hold, then any solution of (2.6) which satisfies (2.2), (2.3), and (2.21) with $V = \mathbf{U}$ for *all* $R \to \infty$ (keep in mind (2.18)) is linearly non–degenerate if $R$ is sufficiently large.

*Proof.* We remark that in the case where $W''(\mu) > 0$, we have already seen in Remark 2.5 that any such solution is in fact asymptotically stable for large $R$.

To prove this proposition, we will argue once more by contradiction. Suppose that there exists a sequence $R \to \infty$ and solutions $u_R$ of (2.6), as in the assertion of the proposition, such that there are nontrivial solutions $\varphi_R$ of (2.35) with $\mu_R = 0$. Without loss of generality, we may assume that the normalization (2.36) holds. This time, the $\varphi_R$'s may change sign but they are still radially symmetric (see again [151], [181], and note that (2.21) implies that $u_R'(R) < 0$ for large $R$). By Lemma 2.1 in [167], the following identity holds

$$R^{-n} \int_0^R W'(u_R(r)) \varphi_R(r) r^{n-1} dr = -\frac{1}{2} u_R'(R) \varphi_R'(R). \tag{2.61}$$

In order to make the presentation as self-contained as possible, let us mention that a direct proof of (2.61) can be given by observing that the function

$$\zeta(r) = r^n \left[u'\varphi' - W'(u)\varphi\right] + (n-2) r^{n-1} u'\varphi, \quad r \in [0, R],$$

(having dropped the subscripts for the moment), introduced in [239], satisfies

$$\zeta'(r) = -2W'(u)\varphi r^{n-1}, \quad r \in (0, R);$$

see also Chapter 1 in [169] (a perhaps simpler proof was given in [168]). Since $W'(\mu) = 0$, by (2.3), we deduce that

$$R^{-n}\int_0^R \left[W'\left(u_R(r)\right)\right]^2 r^{n-1}dr \to 0 \ \text{ as } R \to \infty.$$

Hence, recalling (2.36), via the Cauchy–Schwarz inequality, we find that the lefthand side of (2.61) tends to zero as $R \to \infty$ (along the sequence). On the other side, from our assumption that (2.21), with $V = \mathbf{U}$, holds for all $R \to \infty$, we know that

$$u_R'(R) \to -\sqrt{2W(0)} < 0 \ \text{ as } \ R \to \infty.$$

So, from (2.61), we get that $\varphi_R'(R) \to 0$ as $R \to \infty$ (along the contradicting sequence). By the continuous dependence theory for systems of ordinary differential equations [27, 247] (applied to $\varphi_R(R-r)$ in (2.35)), making use of (2.21) with $V = \mathbf{U}$ for all $R \to \infty$, we infer that for any $D > 0$ we have

$$|\varphi_R(r)| + |\varphi_R'(r)| \leq \frac{1}{2}, \quad r \in [R-D, R], \tag{2.62}$$

provided that $R$ is sufficiently large (along this sequence). On the other hand, if $D$ is chosen so that $W''(u_R) \geq 0$ on $\bar{B}_{(R-D)}$, which is possible by (1.15), (2.2) and (2.3), it follows from (2.35) with $\mu_R = 0$ that

$$\varphi_R \Delta \varphi_R = W''(u_R)\varphi_R^2 \geq 0 \ \text{ on } \bar{B}_{(R-D)},$$

for such large $R$. In particular, we find that $\varphi_R$ cannot vanish in $B_{(R-D)}\backslash\{0\}$ (using the radial symmetry, and integrating by parts over $B_z$ if $\varphi_R(z) = 0$). Furthermore, it cannot vanish at the origin by virtue of the uniqueness theorem for ordinary differential equations, which still holds despite of the singularity at $r = 0$ (see [99], [205], [247]). Therefore, without loss of generality, we may assume that $\varphi_R > 0$ in $B_{(R-D)}$. Hence, the positive function $\varphi_R$ is subharmonic in $B_{(R-D)}$, and not greater than $\frac{1}{2}$ on $\partial B_{(R-D)}$ (recall (2.62)), for $R$ large along the contradicting sequence. The maximum principle (see for example Theorem 2.3 in [142]) yields that $0 < \varphi_R \leq \frac{1}{2}$ on $\bar{B}_{(R-D)}$. The latter relation together with (2.62) clearly contradict (2.36), and we are done.

The proof of the proposition is complete. □

**Remark 2.14.** We note that identity (2.61) has been generalized in Lemma 2.3 in [201] for the case of solutions of (1.2) on an arbitrary smooth, bounded star-shaped domain (see also Theorem 1.6 in [169]). This leaves open the possibility that Proposition 2.1 above can be generalized accordingly.

The following corollary is a simple consequence of the maximum principle.

**Corollary 2.1.** If $W''(\mu) > 0$, then the solutions provided by Lemmas 2.1 and 2.3 satisfy

$$\mu - u_R(r) \leq C_5 e^{-C_6(R-r)}, \quad r \in [0, R-2D] \ \text{ for } \ R \geq R',$$

and some positive constants $C_5, C_6$, depending on $W$ and $n$.

*Proof.* Let $\varphi \equiv \mu - u_R$, where $u_R$ is as in Lemma 2.1 or 2.3. By virtue of **(a')**, (2.2), and (2.3), we can choose $\epsilon$ sufficiently small such that that

$$W'(u_R(x)) \leq \frac{W''(\mu)}{2}\left(u_R(x) - \mu\right), \quad x \in B_{(R-D)},$$

provided that $R \geq R'$, where $D$, $R'$ are as in the previously mentioned lemmas (having increased the value of $R'$, if necessary, but still depending only on $\epsilon$, $D$, $W$, and $n$). It follows from (2.6) that

$$-\Delta\varphi + \frac{W''(\mu)}{2}\varphi \leq 0 \ \text{ in } B_{(R-D)}, \ R \geq R'.$$

Now, the desired assertion of the corollary follows from a standard comparison argument, see Lemma 2 in [49] or Lemma 4.2 in [129] (see also Lemma 2.5 in [112] and Lemma 5.3 in [138]).

The proof of the corollary is complete. □

**Remark 2.15.** A special case of Theorem 2.1 in [93] shows that the assertion of Corollary 2.1 above can be considerably refined to

$$\lim_{R\to\infty} R^{-1}\ln\left(\mu - u_R(Rs)\right) = -(1-s)\sqrt{W''(\mu)}, \quad \forall\, s \in [0,1],$$

see also [30].

2.2. **Proof of Theorem 1.2.** Once Lemma 2.1 is established, the proof of Theorem 1.2 proceeds in a rather standard way. We will present two different approaches, and leave it to the reader's personal taste. The first approach is based on the method of upper and lower solutions, while the second one is based on variational arguments.

`First proof of Theorem 1.2`: We will adapt an argument from the proof of Theorem 2.1 in [98], and prove existence of the desired solution to (1.2) by the method of upper and lower solutions (see for instance [188], [217]). Let $\epsilon \in (0,\mu)$ and $D > D'$, where $D'$ is as in (1.11), and $R'$ be the positive constant, depending only on $\epsilon$, $D$, $W$, and $n$, that is described in Lemma 2.1. Suppose that $\Omega$ contains a closed ball of radius $R'$. We use $\bar{u}(x) \equiv \mu,\ x \in \Omega$, as an upper solution (recall that $W'(\mu) = 0$), and as lower solution the function

$$\underline{u}_P(x) \equiv \begin{cases} u_{\mathrm{dist}(P,\partial\Omega)}(x-P), & x \in B_{\mathrm{dist}(P,\partial\Omega)}(P), \\ 0, & x \in \Omega \backslash B_{\mathrm{dist}(P,\partial\Omega)}(P), \end{cases} \tag{2.63}$$

for some $P \in \Omega_{R'}$ (considered fixed for now), where $u_R$ is as in Lemma 2.1 (here we used that $W'(0) \leq 0$ and Proposition 1 in [35] to make sure that $\underline{u}_P$ is a lower solution, see also Proposition 1 in [173]). In view of (2.2) and (2.3), keeping in mind that

$$\mathrm{dist}(P,\partial\Omega) > R', \tag{2.64}$$

it follows that

$$\underline{u}_P(x) < \bar{u}(x) \equiv \mu, \ x \in \Omega, \ \text{ and } \ \mu - \epsilon < \underline{u}_P(x), \ \ x \in B_{(\mathrm{dist}(P,\partial\Omega)-D)}(P). \tag{2.65}$$

In the case where $\Omega$ is bounded, it follows immediately from the method of monotone iterations, see Theorem 2.3.1 in [217] (this is the only place where we use the smoothness of $\partial\Omega$, see however Remark 2.16 below), that there exists a solution $u \in C^2(\Omega)\cap C(\bar{\Omega})$ of (1.2) such that

$$\underline{u}_P(x) < u(x) < \bar{u}(x) \equiv \mu, \quad x \in \Omega, \tag{2.66}$$

(keep in mind that the solution $u$ depends on the choice of the center $P$). The same property can also be shown in the case where $\Omega$ is unbounded, by exhausting it with a sequence of bounded domains, see Theorem 2.10 in [193] (also recall our discussion following the statement of Theorem 1.2), see also [197, 199]. We have thus established the existence of a solution $u$ to (1.2) that satisfies (1.3), and the lower bound (1.13) in the region $B_{(\mathrm{dist}(P,\partial\Omega)-D)}(P)$ (recall (2.2), (2.3), and (2.64)), or equivalently in $P+B_{(\mathrm{dist}(P,\partial\Omega)-D)} \supseteq P+B_{(R'-D)}$. It remains to show that the latter lower bound is valid in $\Omega_{R'}+B_{(R'-D)}$. To this end, as we will see in a moment, in a more complicated setting, we can translate the compactly supported function $u_{R'}(x-Q)$, for $x \in B_{R'}(Q)$ (extended by zero otherwise), $Q \in \Omega_{R'}$, starting from $Q=P$, by means of Serrin's sweeping technique and the sliding method, to obtain the lower bound (1.13) via (2.3).

In the remainder of this proof, unless specified otherwise, we will assume that relation (1.15) holds. Observe that as we vary the point $P$ in $\Omega_{R'}$, assuming for the moment that $\Omega_{R'}$ has a single arcwise connected component, the functions $\underline{u}_P$'s continue to be lower solutions of (1.2). Consequently, by Serrin's sweeping principle (see [86, 91, 157, 217], and the last part of the proof of Proposition 3.1 herein), we deduce that

$$\underline{u}_Q(x) < u(x), \quad x \in \Omega, \quad \forall\, Q \in \Omega_{R'}, \tag{2.67}$$

(see also the proof of Lemma 3.1 in [86], and note that $\underline{u}_Q$ varies continuously with respect to $Q$ because of the connectedness of $\partial\Omega$; by the implicit function theorem [24], we obtain that $u_R$ varies smoothly with respect to $R$, provided that $R$ is sufficiently large so that Proposition 2.1 is applicable).

In fact, this is more in the spirit of the celebrated sliding method [40]: Let $\gamma(s)$, $s \in [0,1]$, be a smooth curve, lying entirely in $\Omega_{R'}$, such that $\gamma(0)=P$ and $\gamma(1)=Q$ ($Q \in \Omega_{R'}$ arbitrary). It follows from (2.66) that $\underline{u}_{\gamma(0)} < u$ in $\Omega$. We intend to show that

$$\underline{u}_{\gamma(s)} \leq u \text{ in } \Omega \text{ for all } s \in [0,1].$$

Call

$$t_* = \sup\left\{t \in [0,1] \ : \ \underline{u}_{\gamma(s)} \leq u \text{ on } \bar{\Omega} \ \ \forall\, s \in [0,t]\right\}.$$

It is easily seen that

$$\underline{u}_{\gamma(t_*)} \leq u \ \text{ on } \bar{\Omega}.$$

Suppose, to the contrary, that $t_* < 1$. Then, there exists a sequence $t_j > t_*$, satisfying $t_j \to t_*$, and a sequence $x_j \in \bar{\Omega}$, such that

$$\underline{u}_{\gamma(t_j)}(x_j) > u(x_j).$$

Clearly, we have that

$$x_j \in B_{\mathrm{dist}(\gamma(t_j),\partial\Omega)}\left(\gamma(t_j)\right).$$

We may therefore assume that

$$x_j \to x_* \in \bar{B}_{\mathrm{dist}(\gamma(t_*),\partial\Omega)}\left(\gamma(t_*)\right).$$

Furthermore, we obtain that

$$\underline{u}_{\gamma(t_*)}(x_*) = \lim_{j\to\infty} \underline{u}_{\gamma(t_j)}(x_j) \geq \lim_{j\to\infty} u(x_j) = u(x_*).$$

Hence, we get that $\underline{u}_{\gamma(t_*)}(x_*) = u(x_*)$. Since $\underline{u}_{\gamma(t_*)} = 0$ on $\partial B_{\mathrm{dist}(\gamma(t_*),\partial\Omega)}\left(\gamma(t_*)\right)$, while $u>0$ in $\Omega$, we infer that $x_* \in B_{\mathrm{dist}(\gamma(t_*),\partial\Omega)}\left(\gamma(t_*)\right)$ or $x_* \in \partial\Omega$, and is a point of local minimum for

$u-\underline{u}_{\gamma(t_*)}$. But, we have

$$\Delta\left(u-\underline{u}_{\gamma(t)}\right)+q_t(x)\left(u-\underline{u}_{\gamma(t)}\right)\leq 0 \ \text{ weakly in } \Omega, \text{ with } q_t\in L^\infty(\Omega),\ t\in[0,1]. \qquad (2.68)$$

If $x_*\in B_{\text{dist}(\gamma(t_*),\partial\Omega)}\left(\gamma(t_*)\right)$, the strong maximum principle implies that $u\equiv u_{\gamma(t_*)}$ therein. However, this is not possible, since $u>u_{\gamma(t_*)}=0$ at points on $\partial B_{\text{dist}(\gamma(t_*),\partial\Omega)}\left(\gamma(t_*)\right)$ which are not on $\partial\Omega$. In passing, we remark that a similar argument, in the case where the radius of the sliding ball is kept fixed, appears in the proof of Lemma 3.1 in [38]. It remains to consider the case where $x_*\in\partial\Omega$, and $u>\underline{u}_{\gamma(t_*)}$ in $\Omega$. For simplicity, we will assume that $u$ and $\underline{u}_{\gamma(t_*)}$ touch only at $x_*$, since the general case can be treated analogously. Given $\rho>0$, there exist $\delta,d>0$ such that

$$u-\underline{u}_{\gamma(t_*)}\geq d \ \text{ on } \bar{B}_{(\text{dist}(\gamma(t_*),\partial\Omega)+\delta)}\left(\gamma(t_*)\right)\cap\bar{\Omega}\setminus B_\rho(x_*),$$

(by the imposed regularity on $\partial\Omega$, we may assume that $B_{(\text{dist}(\gamma(t_*),\partial\Omega)+\delta)}\left(\gamma(t_*)\right)\cap\Omega$ and $B_\rho(x_*)\cap\Omega$ contain only one connected component respectively). Hence, for $t$ close to $t_*$ (how close depends on the smallness of $\rho>0$), we have

$$u-\underline{u}_{\gamma(t)}\geq 0 \ \text{ on } \bar{B}_{\text{dist}(\gamma(t),\partial\Omega)}\left(\gamma(t)\right)\setminus B_\rho(x_*).$$

In turn, the latter relation clearly implies that

$$u-\underline{u}_{\gamma(t)}\geq 0 \ \text{ on } \bar{\Omega}\setminus B_\rho(x_*).$$

In particular, we find that $u-\underline{u}_{\gamma(t)}\geq 0$ on the boundary of $B_\rho(x_*)\cap\Omega$ for $t-t_*>0$ small. Decreasing the value of $\rho>0$, if necessary, we can apply the maximum principle for small domains in (2.68) (see [61], [72], [92]), to infer that $u-\underline{u}_{\gamma(t)}\geq 0$ in $B_\rho(x_*)\cap\Omega$. We have thus arrived at $u-\underline{u}_{\gamma(t)}\geq 0$ on $\bar{\Omega}$ for $t-t_*>0$ sufficiently small (depending on the smallness of $\rho>0$), which contradicts the assumption that $t_*<1$. Consequently, we have that $t_*=1$, as desired.

The validity of the lower bound (1.13), over the whole specified domain, now follows from (2.3), (2.63), (2.64), and (2.67). In the case where the domain $\Omega_{R'}$ has numerably many arcwise connected components, we can use the function $\max\{\underline{u}_{P_i},\ i=1,\cdots\}$ as a lower solution, where the $\underline{u}_{P_i}$'s are as in (2.63) with each center $P_i$ belonging to a different component of $\Omega_{R'}$. (We use again Proposition 1 in [35], keep in mind that the maximum is essentially chosen among finitely many functions). The case where there are denumerably many arcwise connected components of $\Omega_{R'}$ can be treated similarly. The proof of (1.14), which does not require assumption (1.15), is postponed until the second proof of Theorem 1.2 that follows.

If $W''(\mu)>0$, the validity of (1.5) for $x\in\Omega_{R'}$ follows at once from Corollary 2.1 and relations (2.63), (2.67). If $\text{dist}(x,\partial\Omega)\leq R'$, then plainly observe that

$$\mu-u(x)\leq\mu=\mu e^{R'}e^{-R'}\leq\mu e^{R'}e^{-\text{dist}(x,\partial\Omega)}. \qquad (2.69)$$

If relation (1.15) holds, then the validity of (1.16) follows from (2.60), (2.63), and (2.67), keeping in mind that $\mu-\epsilon\leq\underline{u}_P(P)\leq u(P)$, via (1.15), implies that $-W'\left(u(P)\right)\leq -W'\left(\underline{u}_P(P)\right)$. We postpone the proof of relation (1.18) until Subsection 4.1.

The first proof of the theorem, with the exception of (1.18), is complete. ☐

**Remark 2.16.** It is stated in page 1107 of [38], unfortunately without providing a reference, that the method of upper and lower solutions works also in the case of merely Lipschitz domains (at least for Dirichlet boundary conditions). If this is true, then our Theorem 1.2 holds for $\Omega$ Lipschitz.

**Remark 2.17.** Since it is constructed by the method of upper and lower solutions, we know that the obtained solution $u$ is stable (with respect to the corresponding parabolic dynamics), see [188, 217], namely the principal eigenvalue of

$$-\Delta\varphi + W''(u)\varphi = \lambda\varphi,\ x \in \Omega;\quad \varphi = 0,\ x \in \partial\Omega,$$

in nonnegative. In the case of unbounded domains, some extra care is needed in the definition of stability, see [68, 97, 113].

`Second proof of Theorem 1.2`: Assume first that $\Omega$ is bounded. As in the proof of Lemma 2.1, there exists a global minimizer $u_{min}$ of the energy

$$J(v;\Omega) = \int_\Omega \left\{\frac{1}{2}|\nabla v|^2 + W(v)\right\} dx,\quad v \in\ W_0^{1,2}(\Omega),$$

which furnishes a classical solution of (1.2) such that $0 \le u_{min}(x) < \mu$, $x \in \Omega$. Again, by the strong maximum principle, either $u_{min}$ is identically equal to zero or it is strictly positive in $\Omega$. We intend to show that there exists an $R_* > 0$, depending only on $W$ and $n$, such that $u_{min}$ is nontrivial, provided that $\Omega$ contains some closed ball of radius $R_*$.

For the sake of our argument, suppose that $u_{min}$ is the trivial solution. Then, motivated from Proposition 1 in [2] (see also [69], [175] and [198]), assuming without loss of generality that $\bar{B}_{R+2} \subset \Omega$ for some $R > 0$, we consider the function

$$Z(x) = \begin{cases} 0, & x \in \Omega\backslash B_{R+1}, \\ \mu(R+1-|x|), & x \in B_{R+1}\backslash B_R, \\ \mu, & x \in B_R. \end{cases} \tag{2.70}$$

Since $Z \in W_0^{1,2}(\Omega)$, from the relation $J(0;\Omega) \le J(Z;\Omega)$, and recalling that $W(\mu) = 0$, we obtain that

$$J(0;B_{R+1}) \le \int_{B_{R+1}\backslash B_R} \left\{\frac{1}{2}|\nabla Z|^2 + W(Z)\right\} dx \le C_0 R^{n-1}, \tag{2.71}$$

with $C_0$ depending only on $W$ and $n$. In turn, the above relation implies that

$$|B_{R+1}|W(0) \le C_0 R^{n-1},$$

which cannot hold if $R \ge R_*$ is sufficiently large, depending on $W$ and $n$. Consequently, the minimizer $u_{min}$ is nontrivial, provided that $\Omega$ contains some closed ball of radius $R_*$. From our previous discussion, we therefore conclude that $u_{min}$ satisfies (1.3).

Let $\epsilon \in (0,\mu)$ and $D > D'$, where $D'$ is as in (1.11). Suppose that $\Omega$ contains a closed ball of radius $R'$, where $R'$ is as in the assertion of Lemma 2.1; without loss of generality, we may assume that $R' > R_*$. Relation (1.13) now follows by applying Lemma A.3 below, over every closed ball of radius $R'$ contained in $\Omega$, and recalling Lemma 2.1. Note that, as in Remark 2.3, the unique continuation principle implies that

$$u_{min} \text{ minimizes } J(v;\mathcal{D}) \text{ in } v - u_{min} \in W_0^{1,2}(\mathcal{D}) \text{ for every smooth bounded domain } \mathcal{D} \subset \Omega. \tag{2.72}$$

The case where $\Omega$ is unbounded can be treated by exhausting it by an infinite sequence of bounded ones, where the above considerations apply (see also [198]). The minimizers over the bounded domains (extended by zero outside) converge locally uniformly to a solution $u$ of (1.2) that satisfies (1.3) (the latter solution is nontrivial by virtue of the lower bound

$u(x) \geq \mu - \epsilon,\ x \in B_{(R'-D)}(x_0)$ for some $x_0 \in \Omega_{R'}$, which is valid since we may assume that each one of the bounded domains contains the same closed ball $\bar{B}_{R'}(x_0)$). This solution of (1.2), on the unbounded domain $\Omega$, found in this way, may have infinite energy but is still a global minimizer in the sense of Definition 1.2 in [159], namely satisfies (2.72). As before, it satisfies (1.13).

The validity of (1.14) follows from (2.4) and Lemma A.3 below (applied on every ball $B_{\mathrm{dist}(x,\partial\Omega)}(x),\ x \in \Omega_{R'}$). Similarly, if $W''(\mu) > 0$, the validity of (1.5) follows from Corollary 2.1, Lemma A.3 below, and the observation in (2.69). The validity of relation (1.16) follows in the same manner, making use of (2.60). We postpone the proof of relation (1.18) until Subsection 4.1.

The second proof of the theorem, with the exception of (1.18), is complete. □

**Remark 2.18.** If $W$ is as in Remark 2.5, and $\Omega$ is bounded with smooth boundary (at least $C^3$), in view of the latter remark and Theorem 2 in [238], the solutions of Theorem 1.2 that we found by the two different approaches are actually the same, if $\epsilon$ is chosen sufficiently small.

**Remark 2.19.** The first proof of Theorem 1.2 provides the additional information of the existence of a minimal and maximal solution of (1.2).

**Remark 2.20.** Assume that the domain $\Omega$ is symmetric with respect to the hyperplane $x_i = 0$. Then, since the solution of (1.2), provided by the second proof of Theorem 1.2, is a global minimizer of the associated energy (in the sense described above, in case $\Omega$ is unbounded), it follows from Theorem II.5 in [185] (applied on symmetric bounded domains, with respect to the hyperplane $x_i = 0$, exhausting $\Omega$) that the latter solution is symmetric with respect to this hyperplane. Note that, if in addition the domain $\Omega$ is bounded and convex in the $x_i$ direction, this assertion holds true for *any* positive solution of (1.2) by virtue Theorem 2 in [61] or Theorem 1 in [92] (proven by the method of moving planes). Clearly, if uniqueness holds for positive solutions of (1.2) (recall Remark 1.3), these assertions follow at once (see also Remark 1.3 in [134]).

**Remark 2.21.** In the case where $\overline{\Omega}$ is the complement in $\mathbb{R}^n$ of a smooth *convex* domain $\mathcal{D}$, the existence of the desired solution to (1.2) can be proven by noting that the function

$$\underline{u}(x) = \mathbf{U}\left(\mathrm{dist}(x, \partial\mathcal{D})\right), \tag{2.73}$$

with $\mathbf{U}$ as in (1.12), is a lower solution to (1.2). This follows from (1.19), and the property that the distance function $\rho(x) = \mathrm{dist}(x, \partial\mathcal{D})$ satisfies $|\nabla\rho| = 1$ and $\Delta\rho \geq 0$ (see [160] and a related discussion in [218]). Actually, it has been proven recently in [177, 209] that these properties also hold for *mean convex* domains $\mathcal{D}$. Keep in mind that $\bar{u} = \mu$ *is always an upper solution.*

In the case where $\Omega$ is the quarter-plane $\{x_1 > 0,\ x_2 > 0\}$ (recall our discussion in the introduction about saddle solutions), and $W$ *also satisfies (1.20)*, it was observed in [220] that the function

$$\frac{1}{\mu}\mathbf{U}\left(\frac{x_1}{\sqrt{2}}\right)\mathbf{U}\left(\frac{x_2}{\sqrt{2}}\right)$$

is a lower solution to (1.2). We note that if the first $\mathbf{U}$ in the above product is replaced by $u_R$, as provided by Lemma 2.1 with $n = 1$, the resulting function becomes a lower solution to (1.2) in the semi-infinite strip $(-\sqrt{2}R, \sqrt{2}R) \times [0, \infty)$; in this regard, recall our discussion on "tick" saddle solutions. Similarly, we can construct lower solutions in a rectangle (recall the so called "lattice" solutions). Analogous constructions hold in arbitrary dimensions. However, it does not seem likely that one can play this game for the so called "pizza" solutions.

**Remark 2.22.** In the case where $\Omega$ is convex, the function

$$\bar{u}(x) = \mathbf{U}\left(\mathrm{dist}(x, \partial\Omega)\right),$$

is a (weak) upper solution to (1.2) (see the comments following (2.73)). Therefore, if uniqueness of positive stable solutions holds, we can generalize (2.40).

## 3. Uniform estimates for positive solutions without specified boundary conditions

In this section, we will assume conditions **(a')**, (1.8), and (1.9). Under these assumptions, we will establish uniform estimates for solutions of

$$\Delta u = W'(u), \tag{3.1}$$

provided that they are positive and less than $\mu$ over a sufficiently large set. Our motivation comes from Lemmas 3.2–3.3 in [38], Lemma 4.1 in [172], and Lemma 6.1 in [224] (see also Lemma 2.4 in [118] and [166]).

The next proposition and the corollary that follows refine the latter results, pretty much as (2.3) refined (2.12). In particular, the approach that we apply for their proofs will be used crucially in the proof of Theorem 10.1 below.

**Proposition 3.1.** Suppose that $W \in C^3$ satisfies **(a')**, (1.8), and (1.9). Let $\epsilon \in (0, \mu)$ and $D > D'$, where $D'$ is given from (1.11). There exists a positive constant $R'$, depending on $\epsilon$, $D$, $W$, $n$, such that for *any* solution of (3.1) which satisfies

$$0 < u(x) < \mu, \quad x \in \bar{B}_R(P), \ \text{for some } P \in \mathbb{R}^n, \quad \text{and } R \geq R', \tag{3.2}$$

we have

$$u(x) \geq \mu - \epsilon, \quad x \in \bar{B}_{(R-D)}(P). \tag{3.3}$$

If $W'' \geq 0$ on $[\mu - \epsilon, \mu]$, we have that

$$\min\left\{W(t) \ : \ t \in [0, u(x)]\right\} \leq \frac{C}{R - |x - P|}, \quad x \in B_R(P), \tag{3.4}$$

for some positive constant $C$ that depends only on $W$ and $n$, and

$$-W'\left(u(P)\right) \leq \tilde{C}(R - |x - P|)^{-2}, \quad x \in B_R(P), \tag{3.5}$$

for some constant $\tilde{C} > 0$ that depends only on $n$.

*Proof.* Before we go into the proof, let us make some remarks. The point of this proposition is that we do not assume that the solutions under consideration are global minimizers, a case which can be handled similarly to the second proof of Theorem 1.2. The argument that was used for the related results in [38], [172], [224] essentially consists in constructing a family of positive lower solutions of (3.1) of the form $s\varphi_R$, $s > 0$, where $\varphi_R$ is the eigenfunction associated to the principal eigenvalue of the negative Dirichlet Laplacian over a fixed ball $B_R$, and sweeping á la Serrin with respect to $s$ (see also Lemma 2.2 herein, Lemma 3.1 in [26], Lemma 2 in [91], Theorem 2.1 in [132], and Proposition 3.1 in [215]). On the other hand, our argument consists in constructing a family of nonnegative lower solutions of (3.1) from the global minimizing solutions of (2.6) that are provided by Lemma 2.1, and sweeping with respect to the radius of the ball (a similar idea can be found in [99], see also the comments after Proposition 2.2 in [148]). Borrowing an expression from [166], this type of argument may appropriately be called "ballooning" (as opposed to "sliding"). The main advantage of our approach will become clear in Theorem 10.1 below.

Observe that if $u$ solves (2.6), the function $v(y) \equiv u(Ry)$, $y \in B_1$, satisfies

$$\Delta v = R^2 W'(v), \ y \in B_1; \quad v(y) = 0, \ y \in \partial B_1. \tag{3.6}$$

Since $W'(0) \leq 0$ (recall **(a')**), it follows from the results in [167] (see also Chapter 1 in [169]), which are based on the identity (2.61), that solutions of (3.6) lie on smooth curves in the $(R, v)$ "plane", i.e. either solutions of (3.6) can be continued in $R$ or else there are simple turning points (see also [164] for the definitions and functional set up). We will distinguish two cases according to $W'(0)$:

- If $W'(0) < 0$, by a classical global result of Leray and Schauder (see [176] or page 65 in [24]), there exists an unbounded connected branch $\mathcal{C}_+ \subseteq (0, \infty) \times C(\bar{B}_1)$ of positive solutions to (3.6) that meets $(0, 0)$ (see also Lemma 5.1 in [23]). (As we have already discussed, thanks to [140], all positive solutions of (3.6) are radially symmetric and decreasing). In fact, the detailed behavior of that branch as $R \to 0^+$ is described in Theorem 3.2 of [202]. By the strong maximum principle, we deduce that the solutions on $\mathcal{C}_+$ take values strictly less than $\mu$ (by a continuity argument, since they do so for small $R$, see also Lemma 1 in [153]). Thus, the projection of $\mathcal{C}_+$ onto $(0, \infty)$ is unbounded, namely coincides with $(0, \infty)$.
- If $W'(0) = 0$ and $W''(0) < 0$ (recall (1.8)), there is a global connected solution curve $\mathcal{C}_+$ in $(0, \infty) \times C(\bar{B}_1)$, emanating from $(R_c, 0)$, where $R_c$ was defined in (2.41), due to a bifurcation from a simple eigenvalue as $R$ crosses $R_c$ (see [164]). As before, the solutions on that branch are positive and strictly less than $\mu$. It follows readily from Rabinowitz's global bifurcation theorem [212] (see also Chapter 4 in [24]) that the projection of $\mathcal{C}_+$ onto $(0, \infty)$ is an unbounded interval (for this point, which relies on the radial symmetry of solutions, see the appendix in [227] for instance). (Keep in mind that $\mathcal{C}_+$ is bounded away from $R = 0$, as can easily be seen by testing (3.6) by the principal eigenfunction $\varphi_1$ in (2.42) (see Lemma 6.17 in [202]); in fact, if (2.43) holds, the projection of $\mathcal{C}_+$ onto $(0, \infty)$ is $(R_c, \infty)$).

As in [90, 167], we can parameterize smoothly $\mathcal{C}_+$ by $\{(R_\tau, v_\tau), \ \tau \in (0, \tau')\}$, for a maximal interval $(0, \tau') \subseteq (0, \mu)$, where $\tau$ is the maximum of $v_\tau$, namely $v_\tau(0) = \tau$. The functions

$$\mathrm{u}_\tau(r) \equiv v_\tau\left(R_\tau^{-1} r\right), \quad \tau \in (0, \tau'), \ \text{where } r = |x|, \tag{3.7}$$

define a smooth, with respect to $\tau$, family of solutions to (2.6), satisfying (2.2), with $R = R_\tau$. Note that

$$R_\tau \to 0, \ \text{as } \tau \to 0, \ \text{if } W'(0) < 0; \ R_\tau \to R_c, \ \text{as } \tau \to 0, \ \text{if (1.8) holds}, \tag{3.8}$$

and the range of $R_\tau$, $\tau \in (0, \tau')$, covers $(0, \infty)$ and $(R_c, \infty)$ respectively (in the latter case, the covered interval might be $[R_1, \infty)$ with $R_1 < R_c$, but strictly positive as can be seen by testing by the principal eigenfunction). In view of Lemma 2.2, it follows that

$$\tau' = \mu.$$

On the other side, from the definition of $\tau$, we see that

$$\mathrm{u}_\tau \to 0, \ \text{uniformly on } \bar{B}_{R_\tau}, \ \text{as } \tau \to 0. \tag{3.9}$$

Given $\epsilon \in (0,\mu)$ and $D > D'$, where $D'$ as in (1.11), let $R'$ be as in (2.3). Suppose that a solution $u$ of (3.1) satisfies (3.2) for some $\mathtt{R} > R'$ and $P \in \mathbb{R}^n$. The family of functions

$$\underline{\mathrm{u}}_{\tau,P}(x) = \begin{cases} \mathrm{u}_\tau(x-P), & x \in B_{R_\tau}(P), \\ 0, & \text{elsewhere}, \end{cases}$$

are lower solutions to (3.1) in $\mathbb{R}^n$ for all $\tau \in (0,\mu)$ (as the maximum of two lower solutions, recall that $W'(0) \leq 0$, see [35]). Moreover, we have

$$\underline{\mathrm{u}}_{\tau,P}(x) = 0 < u(x), \quad x \in \partial B_{\mathtt{R}}(P), \quad \tau \in (0,\tau_*],$$

where $\tau_*$ is the smallest number such that

$$R_{\tau_*} = \mathtt{R},$$

(such a number exists since $R_\tau$ is smooth and $R_{\tau_i} \to \infty$ for some sequence $\tau_i \to \infty$). Also, thanks to (3.8) and (3.9), we get

$$\underline{\mathrm{u}}_{\tau,P}(x) < u(x), \quad x \in \bar{B}_{\mathtt{R}}(P), \text{ for } \tau \text{ close to } 0.$$

Consequently, by Serrin's sweeping principle (see [86, 91, 217]), we deduce that

$$\underline{\mathrm{u}}_{\tau_*,P}(x) \leq u(x), \quad x \in \bar{B}_{\mathtt{R}}(P).$$

In turn, this implies that

$$u_{\mathtt{R}}(x-P) = \mathrm{u}_{\tau_*}(x-P) = \underline{\mathrm{u}}_{\tau_*,P}(x) \leq u(x), \quad x \in \bar{B}_{\mathtt{R}}(P), \tag{3.10}$$

where $u_{\mathtt{R}}$ is some solution to (2.6) that satisfies (2.2) with $R = \mathtt{R}$. To prove this, we let $\tilde{\tau} = \sup\left\{\tau \in (0,\tau_*] \ : \ u \geq \underline{\mathrm{u}}_{\tau,P} \text{ on } \bar{B}_{\mathtt{R}}(P)\right\}$, note that $u(x) \geq \mathrm{u}_{\tilde{\tau}}(x-P)$, $x \in \bar{B}_{R_{\tilde{\tau}}}(P)$, and apply the strong maximum principle to $u(x) - \mathrm{u}_{\tilde{\tau}}(x-P)$ to deduce that this function has a positive lower bound on $\bar{B}_{R_{\tilde{\tau}}}(P)$ if $\tilde{\tau} < \tau_*$; this implies that the same holds true for the function $u - \underline{\mathrm{u}}_{\tilde{\tau},P}$ on $\bar{B}_{\mathtt{R}}(P)$ which contradicts the maximality of $\tilde{\tau}$ if $\tilde{\tau} < \tau_*$. Relation (3.10), by virtue of Lemma 2.2 (recall that $\mathtt{R} > R'$), clearly implies the validity of (3.3).

If $W'' \geq 0$ on $[\mu-\epsilon,\mu]$, from Remark 7.1 below, we know that (2.6) has a unique solution satisfying (2.2) for large $R$. In particular, the solution $u_{\mathtt{R}}$ in (3.10) is the global minimizer of Lemma 2.1, provided that $\mathtt{R}$ is sufficiently large. The validity of relation (3.4) now follows at once from (2.4) and (3.10). Finally, relation (3.5) follows immediately from (2.60) and (3.10).

The proof of the proposition is complete. □

**Remark 3.1.** In the case where condition (1.20) holds, relation (3.10) follows directly from Serrin's sweeping principle. Indeed, it is easy to verify that the functions $tu_{\mathtt{R}}(x-P)$, $0 \leq t \leq 1$, fashion a family of lower solutions to (2.6) which vanish along $\partial B_{\mathtt{R}}(P)$.

**Remark 3.2.** The assumption (1.8) is essential for our approach. Indeed, if $W'(0) = 0$ and $W''(0) = 0$ then there are no arbitrarily uniformly small positive solutions of (3.6) for any $R > 0$ (thanks to the implicit function theorem, see for example [164]). In fact, for the case $W'(t) = rt^p - t^q$, $t \geq 0$, with $p > q > 1$, $r > 0$, which satisfies **(a')**, (1.9), and (1.15) but not (1.8), the global bifurcation diagram of positive solutions of (3.6) has been shown in [201] to be qualitatively the same as the one corresponding to (1.10) that we described at the end of Remark 2.6, namely $\subset$-shaped. It might also be useful to see the condition on the behavior of $W'$ near the origin for the so-called "hair–trigger effect" to take place in the parabolic equation $u_t = \Delta u - W'(u)$, see [28].

**Remark 3.3.** In [38], the assumption (1.8) was replaced by the weaker one: $W'$ being Lipschitz continuous and $-W'(t) \geq \delta_0 t$ on $[0, t_0]$ for some $\delta_0, t_0 > 0$. A possible "cure" for this could be the use of bifurcation theory for Hölder continuous nonlinearities (see Appendix B in [202] and the references therein).

**Remark 3.4.** The proof of the above proposition may be adapted to provide an alternative proof of Lemma 3.1 in [26]. Therefore, one can estimate the width of the boundary layer of positive solutions to the spatially inhomogeneous singular perturbation problem (8.5) below (with $W(\cdot, x)$ essentially satisfying the assumptions of this section for each $x$) only in terms of $W$, without involving the principal Dirichlet eigenvalue of the Laplacian in the unit ball of $\mathbb{R}^n$ (see also Remark 8.6 below).

The following corollary can be deduced from Proposition 3.1 by making use of the celebrated sliding method.

**Corollary 3.1.** Suppose that $W \in C^3$ satisfies **(a')** and (1.9). Let $\epsilon \in (0, \mu)$ and $D > D'$, where $D'$ is given from (1.11). There exists an $R' > D$, depending only on $\epsilon$, $D$, $W$, and $n$, such that any solution $u$ of (3.1) which satisfies (1.3) in a domain $\Omega \neq \mathbb{R}^n$ (open and connected set), containing some closed ball of radius $R'$, satisfies (1.13), (1.14), and (1.16). In the case where $\Omega = \mathbb{R}^n$, the only solutions of (1.22) such that $0 \leq u(x) \leq \mu$, $x \in \mathbb{R}^n$, are the constant ones, namely $u \equiv 0$ and $u \equiv \mu$.

*Proof.* Suppose that $\epsilon$, $D$, $\Omega \neq \mathbb{R}^n$, $u$ are as in the first assertion of the corollary. From our assumptions, we know that $\Omega$ contains some closed ball $\bar{B}_{\mathtt{R}}(P)$ for some $\mathtt{R} \geq R'$ and $P \in \Omega_{\mathtt{R}}$. Since $u$ satisfies (1.3) and (3.1) in $\Omega$, it follows from the proof of Proposition 3.1 that relation (3.10) holds. As in the first proof of Theorem 1.2, we can use the sliding method to show that the latter relation actually holds for all $P \in \Omega_{\mathtt{R}}$. We point out that here we do not need that the boundary of $\Omega$ is continuous, since the radius of the ball is fixed, and we can apply directly Lemma 3.1 in [38]. The validity of the first assertion of (1.13) now follows at once form (2.3) (keep in mind that $\mathtt{R}$ could also be chosen as $R'$). Now, let $Q \in \Omega_{R'} + B_{(R'-D)}$. From the proof of Proposition 3.1, using Serrin's sweeping principle, we have that $u(x) \geq u_{\text{dist}(Q,\partial\Omega)}(x - Q)$ in $B_{\text{dist}(Q,\partial\Omega)}(Q)$. By means of (2.4) and (3.5), we infer that $u$ also satisfies (1.14) and (1.16) respectively. Consequently, we have established the first assertion of the corollary.

The second assertion of the corollary follows easily. By the strong maximum principle, we deduce that either $u \equiv 0$, $u \equiv \mu$, or $0 < u(x) < \mu$, $x \in \mathbb{R}^n$. We will show that the latter alternative cannot happen. Suppose to the contrary that $0 < u(x) < \mu$, $x \in \mathbb{R}^n$. Then, we get that (3.10) holds for every $\mathtt{R} > 0$ and $P \in \mathbb{R}^n$. By fixing $P$ and letting $\mathtt{R} \to \infty$, making use of (2.12), we obtain that $u \geq \mu$ in $\mathbb{R}^n$; a contradiction.

The proof of the corollary is complete. $\square$

**Remark 3.5.** The second assertion of the corollary is a Liouville type theorem, and was originally proven in [28] by parabolic methods (see also [43] for a simpler proof of a more general result, using elliptic techniques, in the spirit of [38]; see also Theorem 2.7 in [114]).

**Remark 3.6.** If we additionally assume that $W''(\mu) > 0$, then Proposition 3.1 and Corollary 3.1 can be derived from the exponential decay estimates of Lemma 4.2 in [172], Proposition 1 in [180], and Lemma 6.2 in [224] (see also [137] and Lemma 2.4 in [147]).

**Remark 3.7.** Estimate (1.16) represents a slight improvement over estimate (3.3) in [38] (see also relation (4.11) below). We remark that the latter relation was shown in [38] without making use of (1.15).

## 4. Algebraic singularity decay estimates in the case of pure power nonlinearity, and completion of the proof of Theorem 1.2

The potential that comes first to mind when looking at **(a')** is

$$W(t) = |t-\mu|^{p+1}, \tag{4.1}$$

where $p \geq 1$.

If $p=1$, the solutions provided by Theorem 1.2 satisfy the exponential decay estimate (1.5). In this section, we will show that a universal algebraic decay estimate holds for *all* solutions of (3.1) with potential as in (4.1), provided that $p>1$. Although our arguments in this section rely on the specific form of the potential $W$, our results may be used together with a comparison argument to cover a broader class of potentials. In particular, as we will show in the following subsection, we can establish the remaining relation (1.18) from the proof of Theorem 1.2. Moreover, our main estimate in this section suggests that there is room for improvement over a result of the celebrated paper [38] by Berestycki, Caffarelli and Nirenberg, see Remark 4.3 below. We believe that the results of this section can have applications in the study of elliptic singular perturbation problems of the form (8.5) below in the case where the degenerate equation $W(u,x)=0$ has a root $u=u_0(x)$ of finite multiplicity (see the recent papers [242, 243, 244]). This section is self-contained and can be studied independently of the rest of the paper.

The main result of this section is

**Proposition 4.1.** Let $W$ be given from (4.1), with $p>1$, and let $\Omega \neq \mathbb{R}^n$ be a domain of $\mathbb{R}^n$. There exists a positive constant $C$, depending only on $p$, $n$, such that *any* solution $u$ of (3.1) in $\Omega$ satisfies

$$|\mu-u| + |\nabla u|^{\frac{2}{p+1}} \leq C\mathrm{dist}^{-\frac{2}{p-1}}(x,\partial\Omega), \quad x\in\Omega. \tag{4.2}$$

In particular, if $\Omega$ is an exterior domain, i.e., $\Omega \supset \{x\in\mathbb{R}^n \ : \ |x|>R\}$ for some $R>0$, then

$$|\mu-u| + |\nabla u|^{\frac{2}{p+1}} \leq C|x|^{-\frac{2}{p-1}}, \quad |x|\geq 2R. \tag{4.3}$$

*Proof.* Our proof is modeled after that of Theorem 2.3 in [206] which dealt with focusing nonlinearities, making use of scaling (blow-up) arguments, inspired from [141], combined with a key "doubling" estimate. The main difference with [206] is in the Liouville type theorem that we will use to conclude, see Remark 4.2 below.

We will argue by contradiction. Suppose that estimate (4.2) fails. Then, there exist sequences of domains $\Omega_k \neq \mathbb{R}^n$, functions $u_k$, and points $y_k\in\Omega_k$, such that $u_k$ solves (3.1) in $\Omega_k$ and the functions

$$M_k \equiv |\mu-u_k|^{\frac{p-1}{2}} + |\nabla u_k|^{\frac{p-1}{p+1}}, \quad k=1,2,\cdots,$$

satisfy

$$M_k(y_k) > 2k\mathrm{dist}^{-1}(y_k,\partial\Omega_k), \quad k=1,2,\cdots.$$

From the *Doubling Lemma* of Polacik, Quittner and Souplet, see Lemma 5.1 and Remark 5.2 (b) in [206] or Lemma C.1 in the appendix below, it follows that there exist $x_k\in\Omega_k$

such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k\mathrm{dist}^{-1}(x_k, \partial\Omega_k), \tag{4.4}$$

and

$$M_k(z) \leq 2M_k(x_k), \quad |z - x_k| \leq kM_k^{-1}(x_k), \quad k = 1, 2, \cdots . \tag{4.5}$$

Note that $B_{kM_k^{-1}(x_k)}(x_k) \subset \Omega_k$. Now, we rescale $u_k$ by setting

$$v_k(y) \equiv \lambda_k^{\frac{2}{p-1}} \left[\mu - u_k(x_k + \lambda_k y)\right], \quad |y| \leq k, \quad \text{with } \lambda_k = M_k^{-1}(x_k). \tag{4.6}$$

The function $v_k$ solves

$$\Delta v_k(y) = (p+1)v_k(y) \left|v_k(y)\right|^{p-1}, \quad |y| \leq k.$$

Moreover,

$$\left[|v_k|^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}}\right](0) = \lambda_k M_k(x_k) = 1, \tag{4.7}$$

and

$$\left[|v_k|^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}}\right](y) \leq 2, \quad |y| \leq k.$$

By using elliptic $L^q$ estimates and standard imbeddings (see [142]), we deduce that some subsequence of $v_k$ converges in $C^1_{loc}(\mathbb{R}^n)$ to a (classical) solution $\mathbf{V}$ of

$$\Delta v = (p+1)v(y) \left|v(y)\right|^{p-1}, \quad y \in \mathbb{R}^n. \tag{4.8}$$

Moreover, thanks to (4.7), we have

$$\left[|\mathbf{V}|^{\frac{p-1}{2}} + |\nabla \mathbf{V}|^{\frac{p-1}{p+1}}\right](0) = 1,$$

so that $\mathbf{V}$ is nontrivial. On the other hand, by a result of Brezis [58], we know that there does not exist a nontrivial solution of (4.8) in $L^p_{loc}(\mathbb{R}^n)$ (in the sense of distributions), see also Theorems 4.6–4.7 in [121] or Theorem B.1 below. Consequently, we have arrived at the desired contradiction.

The proof of the proposition is complete. □

**Remark 4.1.** The powers $2/(p-1)$ and $(p+1)/(p-1)$ in (4.2) (for $u$ and $|\nabla u|$ respectively) are sharp for $p \in \left(1, \frac{n}{n-2}\right)$ if $n \geq 3$ and $p > 1$ if $n = 2$, as can be seen from the explicit solution $u(x) = c(p,n)|x|^{-\frac{2}{p-1}}$ of

$$\Delta u = u^p, \tag{4.9}$$

in $\mathbb{R}^n \backslash \{0\}$, see for example [59].

In the latter reference, it was shown that every nonnegative solution $u \in C^2(B_R)$ of (4.9), with $p > 1$, satisfies

$$u(0) \leq C(p,n)R^{-\frac{2}{p-1}},$$

where $C(p,n)$ is determined explicitly. This result, minus the explicit dependence of the constat on $p, n$, follows as a special case of our Proposition 4.1 if we choose $\mu = 0$. Moreover, it was shown in the same reference that every nonnegative solution $u \in C^2\left(B_R \backslash \{0\}\right)$ of (4.9), with $p \in \left(1, \frac{n}{n-2}\right)$ if $n \geq 3$ and $p > 1$ if $n = 2$, satisfies

$$u(x) \leq l(p,n)|x|^{-\frac{2}{p-1}} \left[1 + \frac{C(p,n)}{l(p,n)} \left(\frac{|x|}{R}\right)^{\beta}\right], \quad 0 < |x| \leq \frac{R}{2},$$

where $\beta = \frac{4}{p-1} + 2 - n > \frac{2}{p-1}$, and $C(p,n), l(p,n)$ some explicitly determined constants. The validity of this estimate, minus the explicit dependence of the constat on $p, n$, follows for *all* the range $p>1$ from our Proposition 4.1 with $\mu=0$. It was shown in [57] that, if $n\geq 3$ and $p\geq \frac{n}{n-2}$, there exists a constant $A=A(p,n)>0$ such that every nonnegative solution $u\in C^2\left(B_1\backslash\{0\}\right)$ of (4.9) satisfies

$$u(x)\leq \frac{A}{|x|^{n-2}},\quad 0<|x|<\frac{1}{2}.$$

In turn, the latter estimate was used to show that the solution $u$ has a removable singularity at the origin. Clearly, the above estimate follows from (4.2) with $\mu=0$. The proofs in [57], [60], and [58] (where we referred to towards the end of the proof of Proposition 4.1), are based on the explicit knowledge of positive, radially symmetric upper solutions of the equation $-\Delta u+|u|^{p-1}u=0$ on arbitrary open balls, with the further property that these functions blow up at the boundary of the considered balls. This fact is crucially related to the shape of the nonlinear function $|t|^{p-1}t$ and it does not easily extend to more general functions. We refer to [121] for a different approach for establishing the Liouville type theorem of [58], that we used towards the end of the proof of Proposition 4.1, with the advantage to apply to a larger class of problems (see Theorem B.1 below).

To the best of our knowledge, this is the first time that the doubling lemma of [206] has been used in relation with the previously mentioned papers.

**Remark 4.2.** In the problems studied in [141], [206] (see also [210]), the blowing-up argument leads to a positive solution of the whole space problem $\Delta v+v^p=0$, which does not exist for the range of exponents $p\in\left(1,\frac{n+2}{n-2}\right)$ if $n\geq 3$, $p>1$ if $n=2$.

**Remark 4.3.** Assume that the potential $W\in C^2$ satisfies (1.9), $W'(t)\geq 0$ for $t\geq\mu$, $-W'(t)\geq \delta_0 t$ on $[0,t_0]$ for some $\delta_0,t_0>0$, and (1.15). Clearly, these conditions are satisfied by the model examples (1.23) and (4.1). Let $\Omega$ be the entire *epigraph*:

$$\Omega=\left\{x\in\mathbb{R}^n\ :\ x_n>\varphi(x_1,\cdots,x_{n-1})\right\},\tag{4.10}$$

where $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ is a globally Lipschitz continuous function. It was shown in Lemma 3.2 in [38] that there are constants $\varepsilon_1,R_0>0$ with $R_0$, depending only on $n,\delta_0$, such that any positive bounded solution of (1.2) satisfies $u<\mu$ in $\Omega$, and

$$u(x)>\varepsilon_1\ \text{ if }\ x\in\Omega_{R_0},\ \text{i.e. }\operatorname{dist}(x,\partial\Omega)>R_0.$$

Moreover, setting

$$\delta(x)=\min\left\{-W'(t)\ :\ t\in[\varepsilon_1,u(x)]\right\},\quad x\in\Omega_{R_0},$$

there exists a constant $C_1$, depending only on $n$, such that

$$C_1\delta(x)\leq\left[\operatorname{dist}(x,\partial\Omega)-R_0\right]^{-2},\quad x\in\Omega_{R_0},\tag{4.11}$$

recall also estimate (1.16) herein. In the case of the potential (4.1), the function $\delta(x)$ is plainly $\delta(x)=(p+1)\left(\mu-u(x)\right)^p$, and estimate (4.11) says that

$$\mu-u(x)\leq\left[C_1(p+1)\right]^{-\frac{1}{p}}\left[\operatorname{dist}(x,\partial\Omega)-R_0\right]^{-\frac{2}{p}},\quad x\in\Omega_{R_0}.$$

Observe that our estimate (4.2) is an improvement of the above estimate, since $\frac{2}{p-1}>\frac{2}{p}$. Moreover, our estimate holds for *all* solutions, possibly sign changing or unbounded, without specified boundary conditions. Note also that these observations reveal that estimate (1.14) is far from optimal.

As in Theorem 2.1 in [206], we can generalize our Proposition 4.1 to

**Proposition 4.2.** Let $p > 1$, and assume that the smooth $W$ satisfies

$$\lim_{|t|\to\infty} t^{-1}|t|^{1-p}W'(t+\mu) = \ell \in (0,\infty). \tag{4.12}$$

Let $\Omega$ be an arbitrary domain of $\mathbb{R}^n$. Then, there exists a constant $C = C(n, W') > 0$ (independent of $\Omega$ and $u$) such that, for any solution of (3.1), there holds

$$|\mu - u| + |\nabla u|^{\frac{2}{p+1}} \le C\left(1 + \mathrm{dist}^{-\frac{2}{p-1}}(x,\partial\Omega)\right), \quad x \in \Omega. \tag{4.13}$$

In particular, if $\Omega = B_R\backslash\{0\}$ then

$$|\mu - u| + |\nabla u|^{\frac{2}{p+1}} \le C\left(1 + |x|^{-\frac{2}{p-1}}\right), \quad 0 < |x| \le \frac{R}{2}.$$

*Proof.* Assume that estimate (4.13) fails. Keeping the same notation as in the proof of Proposition 4.1, we have sequences $\Omega_k,\ u_k,\ y_k \in \Omega_k$ such that $u_k$ solves (3.1) in $\Omega_k$ and

$$M_k(y_k) > 2k\left(1 + \mathrm{dist}^{-1}(y_k, \partial\Omega_k)\right) > 2k\mathrm{dist}^{-1}(y_k,\partial\Omega_k).$$

Then, formulae (4.4)–(4.6) are unchanged but now the function $v_k$ solves

$$\Delta_y v_k(y) = f_k\left(v_k(y)\right) \equiv -\lambda_k^{\frac{2p}{p-1}} W'\left(\mu - \lambda_k^{-\frac{2}{p-1}} v_k(y)\right), \quad |y| \le k.$$

Note that, since $M_k(x_k) \ge M_k(y_k) > 2k$, we also have

$$\lambda_k \to 0, \quad k \to \infty.$$

Since there exists a constant $C > 0$ such that $|W'(\mu - t)| \le C(1 + |t|^p)$, $t \in \mathbb{R}$, due to (4.12) (and $W'$ being continuous), it follows that

$$|f_k(t)| \le C\lambda_k^{\frac{2p}{p-1}} + C|t|^p, \quad t \in \mathbb{R},\ k \ge 1.$$

By using elliptic $L^q$ estimates, standard imbeddings, and (4.12), we deduce that some subsequence of $v_k$ converges in $C^1_{loc}(\mathbb{R}^n)$ to a classical solution $\mathbf{V}$ of $\Delta v = \ell v|v|^{p-1}$ in $\mathbb{R}^n$. Moreover, we have that $|\mathbf{V}(0)|^{\frac{p-1}{2}} + |\nabla\mathbf{V}(0)|^{\frac{p-1}{p+1}} = 1$, so that $\mathbf{V}$ is nontrivial. As in Proposition 4.1, since $\ell > 0$, this contradicts the Liouville theorem in [61], [121], in particular Theorem B.1 below.

The proof of the proposition is complete. □

**Remark 4.4.** The same assertion of Proposition 4.2 holds, if we assume that the righthand side of (4.12) is as the function $f$ in Theorem B.1 below.

4.1. **Proof of relation (1.18).** Based on Proposition 4.1, via a comparison argument, we can show relation (1.18) and thus complete the proof of Theorem 1.2.

`Proof of (1.18):` Clearly, estimate (1.18) holds if $\mathrm{dist}(x,\partial\Omega) \le R'$.

In any connected component $\mathcal{A}$ of $\Omega_{R'} + B_{(R'-D)}$, thanks to (1.13) and (1.17) (assuming that $\epsilon < d$), we have

$$\Delta u \le -c(\mu - u)^p \text{ in } \mathcal{A}, \quad u \ge \mu - \epsilon \text{ on } \partial\mathcal{A}.$$

Let $0 < v < \mu$ be the solution of

$$\Delta v = -c(\mu - v)^p \quad \text{in } \mathcal{A}; \quad v = 0 \quad \text{on } \partial\mathcal{A},$$

as provided by Theorem 1.2 (keep in mind the second part of Remark 1.3 which implies uniqueness), where $c > 0$, $p > 1$ are as in (1.17). From Proposition 4.1, we know that $v$ satisfies

$$\mu - v \leq \hat{K}\mathrm{dist}^{-\frac{2}{p-1}}(x, \partial\mathcal{A}), \quad x \in \mathcal{A},$$

for some constant $\hat{K} > 0$ that depends only on $c, p$, and $n$. Since $\mathrm{dist}(\partial\mathcal{A}, \partial\Omega) \leq D$, we have

$$\mathrm{dist}(x, \partial\mathcal{A}) \geq \mathrm{dist}(x, \partial\Omega) - \mathrm{dist}(\partial\mathcal{A}, \partial\Omega) \geq \mathrm{dist}(x, \partial\Omega) - D.$$

So, we arrive at

$$\mu - v \leq \tilde{K}\mathrm{dist}^{-\frac{2}{p-1}}(x, \partial\Omega), \quad x \in \mathcal{A},$$

for some constant $\tilde{K} > 0$ that depends only on $c, p, n$ and $W$.

We intend to show that

$$v \leq u \ \text{ in } \mathcal{A}, \tag{4.14}$$

from where relation (1.18) follows at once. Let $w = u - v$. We have

$$\Delta w \leq q(x)w \ \text{ in } \mathcal{A},$$

where

$$q(x) = cp\int_0^1 (\mu - su - (1-s)v)^{p-1}\, ds.$$

This is a bit meshy but what matters is that $q$ is continuous and nonnegative in $\mathcal{A}$. Note that $w > 0$ on $\partial\mathcal{A}$ and $w$ is bounded in $\mathcal{A}$ ($|w| \leq \mu$ to be more precise). Therefore, the assumption that $\bar{\Omega}$ is disjoint from the closure of an infinite open connected cone (and so is $\bar{\mathcal{A}}$), or $n = 2$ and $\bar{\Omega} \neq \mathbb{R}^2$, allows us to apply the maximum principle, even in the case where $\mathcal{A}$ is unbounded, to deduce that (4.14) holds (see Lemma 2.1 and the remark following it in [38], and also Lemma 6.2 in [166]).

The proof of relation (1.18) is complete. $\square$

**Remark 4.5.** The proof of Lemma 2.1 in [38] is based on the property that, for the domains $\Omega$ in the previous class, there exists a positive super-harmonic function $g$ in $\Omega$ (i.e. $\Delta g \leq 0$ in $\Omega$) such that $g(x) \to \infty$ if $x \in \Omega$ and $|x| \to \infty$. If $\Omega$ is contained in the set $\{x_n \geq h(x_1, \cdots, x_{n-1}),\ (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}\}$, for some $h \in C(\mathbb{R}^{n-1})$ such that $h(y) \to \infty$ as $|y| \to \infty$, then clearly we can take

$$g(x) = x_n - \min_{\mathbb{R}^{n-1}} h + 1.$$

In light of the comparison function of Theorem 2 in [92], it follows readily that Lemma 2.1 in [38] also applies in the case where $\Omega$ is contained in some strip

$$\{|x_i| \leq M,\ i = 1, \cdots, n-m;\ x_j \in \mathbb{R},\ j = n-m+1, \cdots, n\},$$

where $M > 0$ and $m \in \{1, \cdots, n-1\}$. Therefore, relation (1.18) also holds for such domains.

## 5. Bounds on entire solutions of $\Delta u = W'(u)$

In this subsection, we will assume that the $C^2$ potential $W$ satisfies (4.12) for some $\mu \in \mathbb{R}$, and there exist $\mu_- < \mu_+$ such that

$$W'(\mu_-) = W'(\mu_+) = 0, \quad W'(t) < 0,\ t < \mu_-;\ W'(t) > 0,\ t > \mu_+. \tag{5.1}$$

We will utilize Propositions 3.1 and 4.2, together with the corresponding parabolic flow to (1.22), in order obtain the following result:

**Proposition 5.1.** Under the above assumptions, we have that *every* solution $u \in C^2(\mathbb{R}^n)$ of (1.22), which is not identically equal to $\mu_-$ or $\mu_+$, satisfies

$$\mu_- < u(x) < \mu_+, \quad x \in \mathbb{R}^n. \tag{5.2}$$

*Proof.* From Proposition 4.2 with $\Omega = \mathbb{R}^n$, i.e. $\mathrm{dist}^{-\frac{2}{p-1}}(x, \partial\Omega) = 0 \ \forall\ x \in \mathbb{R}^n$, we know that there exists a constant $C = C(W', n) > 0$ such that every solution of (1.22) satisfies

$$|u(x)| \le C, \quad x \in \mathbb{R}^n.$$

We will show that

$$\mu_- \le u(x) \le \mu_+, \quad x \in \mathbb{R}^n. \tag{5.3}$$

Indeed, as in [26], [109], by the parabolic maximum principle (this is possible since all the functions under consideration are bounded, see [208]), we infer that

$$u_-(t) \le u(x) \le u_+(t), \quad t \ge 0, \tag{5.4}$$

where $u_\pm$ are the solutions of the initial value problems

$$\dot{u}_\pm = W'(u_\pm), \quad t > 0, \quad u_\pm(0) = \pm 2C.$$

Note that $u_\pm(t)$ are solutions of $u_t = \Delta u - W'(u)$ on $\mathbb{R}^n \times (0, \infty)$, as is $u(x)$. From our assumptions on $W$, it follows that $u_-(t)$ is increasing and $u_+(t)$ is decreasing with respect to $t > 0$. Furthermore, it is easy to show that $u_\pm(t) \to \mu_\pm$ as $t \to \infty$, see also [27], [247]. Hence, letting $t \to \infty$ in (5.4), we find that relation (5.3) holds. For a similar argument, which allows for the last assumption in (5.1) to be weakened (allowing $W'$ to vanish), we refer to Theorem 2.7 in [114]. Alternatively, we could argue as in Corollary 3.1 by considering the function $2C - u$. By the strong maximum principle, it follows that (5.2) holds unless $u \equiv \mu_-$ or $u \equiv \mu_+$.

The proof of the proposition is complete. □

**Remark 5.1.** With trivial modifications, Proposition 5.1 can be applied in the case where there is an obstacle in $\mathbb{R}^n$, as in problem (6.1) below (see also Remark 6.4).

As a corollary to the above proposition, we can give a short proof of a Liouville type result in [108] (see Theorem 1.1 therein), where a squeezing argument involving boundary blow-up solutions (recall the discussion related to [58] at the end of Remark 4.1) was used instead (see also [109], [111]).

**Corollary 5.1.** Let $\lambda \in (-\infty, \infty)$, $p > 1$, and $u \in C^2(\mathbb{R}^n)$ be a nonnegative solution of

$$\Delta u = u^p - \lambda u \ \text{ in } \mathbb{R}^n.$$

Then, the solution $u$ must be a constant.

*Proof.* If $\lambda \le 0$, we have that $-\Delta u + u^p \le 0$. Since $p > 1$, it follows from Keller-Osserman theory [163, 200] that $u \le 0$ on $\mathbb{R}^n$ (see Theorem B.1 below). Hence, in this case, the solution $u$ is identically zero.

If $\lambda > 0$, it follows readily from Proposition 5.1 that either $u \equiv 0$ or $u \equiv \lambda^{\frac{1}{p}}$ or $0 < u(x) < \lambda^{\frac{1}{p}}$, $x \in \mathbb{R}^n$. However, the latter alternative cannot occur, because of the second assertion of Corollary 3.1.

The proof of the proposition is complete. □

**Remark 5.2.** Our method of proof, as well as that of [108, 109, 111], work for a broader class of nonlinearities. In the special case of the Allen-Cahn equation

$$\Delta u = u^3 - u \ \text{ in } \mathbb{R}^n, \tag{5.5}$$

by making use of Kato's inequality and Keller-Osserman theory, it was shown in [62] (see also [116], [186]) that all solutions of this equation satisfy $|u(x)| \leq 1$, $x \in \mathbb{R}^n$ (for different proofs, see Lemma 1 in [79] and Lemma 4.1 in [83]). A parabolic version of this result can be found in [186].

The importance of the above results is that they imply that there is no need for the boundedness assumption is the well known statement of the famous De Giorgi's conjecture: *Let $u$ be a bounded solution of equation (5.5) such that $u_{x_n} > 0$. Then the level sets $\{u = \lambda\}$ are all hyperplanes,* ***at least*** *for dimension $n \leq 8$.* The motivation behind this conjecture came from the classical Bernstein problem in the theory of minimal surfaces, which explains the restriction in the dimension. There has been tremendous activity in the last years, and this conjecture has been completely resolved in dimensions $n \leq 3$ (see [137], [22]; see also [190] for an earlier related proof in two dimensions under additional assumptions), and essentially in dimensions $4 \leq n \leq 8$ (assuming that $u \to \pm 1$ pointwise as $x_n \to \pm\infty$, see [218] and also [249]), while a counterexample which shows that the conjecture is false for $n \geq 9$ has been constructed in [102]. For more details, we refer the interested reader to the review article [123] (some more recent proofs in two and three dimensions can also be found in [219]).

## 6. Nonexistence of nonconstant solutions with Neumann boundary conditions

In this section, motivated from a Liouville-type theorem in [45], we will consider some situations where the equation $\Delta u = W'(u)$, in a possibly unbounded domain, with Neumann boundary conditions, has only the (obvious) constant solution.

### 6.1. **A Liouville theorem arising in the study of traveling waves around an obstacle.**

In Theorem 6.1 of their article [45], H. Berestycki, F. Hamel, and H. Matano proved the following Liouville type result:

**Theorem 6.1.** Let $\Omega$ be a smooth, open, connected subset of $\mathbb{R}^n$, $n \geq 2$, with outward unit normal $\nu$, and assume that $K = \mathbb{R}^n \backslash \Omega$ is compact. Let $\mu_- \leq u \leq \mu$ be a classical solution of

$$\begin{cases} \Delta u = W'(u) & \text{in } \Omega, \\ \nu \nabla u = 0 & \text{on } \partial\Omega, \\ u(x) \to \mu & \text{as } |x| \to \infty, \end{cases} \tag{6.1}$$

where $W \in C^2$ satisfies conditions **(a")** (defined prior to Lemma 2.3) with $W(\mu_-) = 0$ allowed, and (1.15). If $K$ is star-shaped, then

$$u \equiv \mu \ \text{ on } \bar{\Omega}. \tag{6.2}$$

In the study conducted in [45] the set $K$ plays the role of an obstacle. The prototypical example for the $W$ in the above theorem is (1.23) (in that case we have $\mu_- = -1$ and $\mu = 1$).

Below, we will provide an alternative proof of the above theorem. We remark that the statement in [45], adapted to our notation, also requires that $W(\mu_-) > 0$ and $W'(\mu_-) = 0$.

In our statement, we assume that $W \in C^2$ in order to apply the implicit function theorem to the equation in (2.6). Nevertheless, with just a slight modification, our proof works also for Lipschitz $W'$ (see Remark 6.2 below), as was the original assumption in [45]. Moreover, as we will see in the same remark, we can easily dispense of assumption (1.15).

Loosely speaking, the approach of [45] consists in using a sweeping family of lower solutions of (6.1), having as building block the solution $\mathbf{U}$ of (1.12). Our proof is in the same spirit, but we build lower solutions out of one dimensional solutions of (2.6), capitalizing on the results of Subsection 2.1. In our opinion, our proof is simpler (having knowledge of Lemma 2.1 and Proposition 2.1 herein) and more intuitive. In particular, our proof of Theorem 6.2 below is considerably simpler than the corresponding one of [45], and does not require that $W'$ is non-decreasing near $\mu$. As will become apparent from the proofs, the main advantage of our approach is that we use lower solutions that stay away from $\mu$ (individually). In contrast, the lower solutions of [45] tend to $\mu$, as $|x| \to \infty$, causing technical difficulties.

`Proof of Theorem 6.1:` Up to a shift of the origin, one can assume without loss of generality that $K$, if not empty, is star-shaped with respect to 0. By the strong maximum principle and Hopf's boundary point lemma [142] (keep in mind that $W'(\mu_-) \leq 0$), and the asymptotic behavior of $u$ as $|x| \to \infty$, we deduce that

$$\inf_{x \in \bar{\Omega}} u(x) > \mu_-.$$

Under our current assumptions on $W$, it is easy to see that analogous assertions to those of Lemma 2.1 hold for minimizers of the energy $J(v; B_R)$ with $v - \mu_- - \delta \in W_0^{1,2}(B_R)$, where $\delta > 0$ is chosen sufficiently small so that $\mu_- + \delta < \inf_{\bar{\Omega}} u$ and $W'(\mu_- + \delta) \leq 0$ (the point is that we have $W(\mu_- + \delta) > 0$; if $W(\mu_-) > 0$ then we can take $\delta = 0$). This is also the case with Proposition 2.1. Abusing notation, we will still denote these minimizers by $u_R$. From Proposition 2.1, there exists an $R_0 > 0$ such that these $u_R$'s with $n = 1$ are non-degenerate for $R \geq R_0$ (abusing notation again). Thus, via the implicit function theorem (see [164]), we can find a continuous family of such minimizing solutions $u_R$ (for $R \geq R_0$, with respect to the uniform topology, as described in Corollary 2.2 in [157]); in this regard, see also Remark 6.6 below. Increasing the value of $R_0$, if necessary, we may assume that

$$W'(u_R(0)) \leq 0, \quad R \geq R_0, \tag{6.3}$$

recall (1.15), $\mu_- + \delta \leq u_R < \mu$, (2.3), and (2.9). By virtue of the asymptotic behavior in (6.1), it follows at once that there exists a large $T > R_0$ such that

$$u(x) > u_{R_0}(0) = \max_{\bar{B}_{R_0}} u_{R_0}, \quad x \in \mathbb{R}^n \backslash B_{(T-R_0)}, \quad \text{and} \quad \bar{K} \subset B_{(T-R_0)}, \tag{6.4}$$

(this is the main advantage of our proof in comparison to [45]). Let

$$\underline{u}_R(r) = \begin{cases} \mu_- + \delta, & r \in (0, \max\{T - R, 0\}), \\ u_R(r - T), & r \in [\max\{T - R, 0\}, T], \\ u_R(0), & r > T, \end{cases} \tag{6.5}$$

with $r=|x|$, $x\in\mathbb{R}^n\backslash\{0\}$. Since $u_R'(0)=0$, it follows that $\underline{u}_R\in C^1(\bar{\Omega})$. Using the equation in (2.6) (with $n=1$), we find that

$$\Delta \underline{u}_R - W'(\underline{u}_R) = \begin{cases} -W'(\mu_- + \delta), & r\in(0,\max\{T-R,0\})\,, \\ \frac{n-1}{r}u_R'(r-T), & r\in(\max\{T-R,0\},T]\,, \\ -W'\left(u_R(0)\right), & r>T. \end{cases} \tag{6.6}$$

In particular, recalling that $W'(\mu_-+\delta)\leq 0$, (2.9) and (6.3), we find that

$$\underline{u}_R \ \text{ is a weak lower solution of (3.1) in } \Omega, \text{ if } R\geq R_0. \tag{6.7}$$

We claim that

$$\underline{u}_R\leq u \ \text{ on } \ \bar{\Omega}, \text{ for all } R\geq R_0. \tag{6.8}$$

Suppose that the claim is false, and let

$$R_*=\sup\left\{R>R_0 \ : \ \underline{u}_s<u \text{ on } \bar{\Omega}, \ s\in(R_0,R)\right\}<\infty,$$

(recall (6.4) which implies that the set of such numbers $s$ is nonempty). The set $\bar{\Omega}$ is not compact, so there need not be a point $x\in\bar{\Omega}$ for which $\underline{u}_{R_*}(x)=u(x)$. However, there exists a sequence of points $x_i\in\bar{\Omega}$ such that $\underline{u}_{R_*}(x_i)-u(x_i)$ tends to zero as $i\to\infty$. Since $u(x)\to\mu$ as $|x|\to\infty$, whereas $\underline{u}_{R_*}(x)\to u_{R_*}(0)<\mu$ as $|x|\to\infty$, it follows at once that the $x_i$'s remain bounded (this is the main advantage of our proof in comparison to [45]). Passing to a subsequence, we find that $x_i\to x_*\in\bar{\Omega}$ with $\underline{u}_{R_*}(x_*)=u(x_*)$. In view of (6.1) and (6.7), we find that

$$\Delta(u-\underline{u}_{R_*})\leq Q(x)(u-\underline{u}_{R_*}) \ \text{ weakly in } \Omega, \tag{6.9}$$

where $Q$ is a continuous function of the form (2.26). Since $u\geq u_{R_*}$ on $\bar{\Omega}$, the weak Harnack inequality (see [142]) tells us that the point $x_*$ must lie on the boundary of $\Omega$ (otherwise, $\underline{u}_{R_*}\equiv u$ which is not possible by (6.6)); note also that at $x_*$ we have that $\underline{u}_{R_*}$ is smooth so we can apply the strong maximum principle. Since $x_*\in\partial\Omega=\partial K$, by (6.9) and Hopf's boundary point lemma, we get that

$$0>\nu\nabla(u-\underline{u}_{R_*})=-\nu\nabla\underline{u}_{R_*}=-(\nu\cdot x_*)\frac{\underline{u}_{R_*}'(|x_*|)}{|x_*|} \ \text{ at } x_*, \tag{6.10}$$

(here $\nu=\nu_{x_*}$ denotes the outward unit normal to $\partial\Omega$ at $x_*$). On the other hand, since $K$ is star-shaped with respect to the origin, we have that

$$x\cdot\nu_x\leq 0, \quad x\in\partial K. \tag{6.11}$$

Also, relation (2.9) implies that

$$\underline{u}_{R_*}'(|x_*|)=u_{R_*}'(|x_*|-T)>0, \quad x\in\mathbb{R}^n\backslash\{0\}.$$

The above two relations contradict (6.10). Consequently, claim (6.8) holds.

Now, letting $R\to\infty$ in (6.8), thanks to (2.3), we arrive at the sought for relation (6.2).

The proof of the theorem is complete. $\square$

**Remark 6.1.** In dimension $n=1$, with $K$ a closed bounded interval, the same arguments can be adapted straightforwardly, and the conclusion of Theorem 6.1 continues to hold. In this special case, however, it is better to use Proposition 6.1 below.

**Remark 6.2.** One can avoid making use of Proposition 2.1 in the proof of Theorem 6.1, and thus have its validity for $W'$ Lipschitz, as follows. Given $\epsilon \in (0, \mu - \mu_-)$ such that $W' \leq 0$ on $[\mu - \epsilon, \mu]$, we can find $R > 0$ and minimizer $u_R$ (as in the above proof) such that $u_R(0) \geq \mu - \epsilon$ (this assertion of Lemma 2.1 holds for $W \in C^{1,1}$). For such $R$ and $u_R$, let $T > R$ be such that (6.4) holds with $R$ in place of $R_0$, namely $u \geq \underline{u}_{R,T}$ $\forall$ $T > R$, where $\underline{u}_{R,T}$ as in (6.5) (with the obvious meaning). Now, in contrast to the proof of Theorem 6.1, we can let $T \to 0$ (perform sliding) and find that $u \geq \mu - \epsilon$ on $\bar{\Omega}$. Since $\epsilon$ can be taken arbitrarily small, we conclude that $u \equiv \mu$, as desired. The same argument can also be applied to Theorem 6.2 below, but does not seem to be usable in Propositions 6.2-6.3 that follow.

Clearly, as $T$ decreases, the functions $u$ and $\underline{u}_{R,T}$ cannot touch at an $x \in \mathbb{R}^n \setminus \bar{B}_T$ (nor at an $x \in \bar{B}_{T-R}$, if $T > R$, as a matter of fact). Hence, there is no need for imposing (1.15) in order to have that $W'(u_R(0)) < 0$ for large $R > 0$. In other words, the assumption (1.15) can also be removed from the statement of Theorem 6.1.

**Remark 6.3.** If we plainly use an $n$-dimensional minimizer from Lemma 2.1 (minimizing over $(\mu_- + \delta) + W_0^{1,2}(B_R)$), making use of Proposition 2.1, and the sliding argument, we can potentially simplify the proof of the related Proposition 2.1 in [54].

**Theorem 6.2.** If in Theorem 6.1 we assume that $N \geq 1$ and the obstacle $K$ to be directionally convex instead of star–shaped, then conclusion (6.2) still holds.

*Proof.* Without loss of generality, we may assume that $K$ is convex in the $x_1$ direction, which implies that

$$(x_1, \cdots, 0)\nu_x \leq 0 \quad \forall\ x = (x_1, \cdots, x_n) \in \partial K, \tag{6.12}$$

where $\nu$ denotes again the unit outer normal to $\partial\Omega$ (i.e. inner to $\partial K$). The proof proceeds along the same lines as that of Theorem 6.1. As in the latter theorem, let $u_R$ denote a minimizing solution to the equation in (2.6) with $n = 1$ and $u_R = \mu_- + \delta$ on $\partial B_R$ (this $\delta > 0$ is completely analogous to the one in the proof of Theorem 6.1). For $R,\ T > 0$, let

$$\underline{u}_R(x) = \begin{cases} u_R(x_1 - T), & x_1 \in \left(\max\{T - R, 0\}, T\right), \\ \\ u_R(x_1 + T), & x_1 \in \left(-T, \min\{-T + R, 0\}\right), \\ \\ u_R(0), & |x| \geq T, \\ \\ \mu_- + \delta, & \text{otherwise.} \end{cases}$$

From the equation in (2.6) and (2.9), we have that $\underline{u}_R$ is a weak lower solution of (3.1) in $\Omega$ (see again [35]). As before, there exist large $R_0,\ T > R_0$ such that $\underline{u}_{R_0} < u$ on $\bar{\Omega}$, and the minimizers $u_R$ vary smoothly with respect to $R \geq R_0$.

We claim that

$$\underline{u}_R \leq u \quad \text{on } \bar{\Omega} \quad \text{for all} \quad R \geq R_0. \tag{6.13}$$

Arguing by contradiction, as in the proof of Theorem 6.1, we get the existence of analogous $R_* > R_0$ and $x_* \in \bar{\Omega}$ (as before, the corresponding sequence $\{x_i\}$ is bounded). To reach a contradiction, it boils down to exclude the case $x_* \in \partial K$. Firstly, note that $x_*$ cannot be on the hyperplane $\{x_1 = 0\}$. Indeed, in that case, we would have $R_* > T$, and observe that the function

$$g(t) = (u - \underline{u}_{R_*})(x_* + te), \quad e = (1, \cdots, 0),$$

would be well defined for small $|t|$ and

$$g'(0^-) - g'(0^+) = u'_{R_*}(-T) - u'_{R_*}(T) = -2u'_{R_*}(T) \overset{(2.9)}{>} 0,$$

which is not possible because $g$ has a global minimum at $t = 0$. Now, since $x_* \in \partial\Omega \backslash \{x_1 = 0\}$, Hopf's boundary point lemma tells us that (6.10) holds. On the other hand, recalling (2.9), at the point $x_*$ we have that

$$(x_1, \cdots, 0)\nabla \underline{u}_{R_*} = x_1 \partial_{x_1} \underline{u}_{R_*} = \begin{cases} x_1 u'_{R_*}(x_1 - T) & \text{if } \ 0 < x_1 < T, \\ \\ x_1 u'_{R_*}(x_1 + T) & \text{if } \ -T < x_1 \leq 0. \end{cases}$$

Hence, relation (2.9) yields that $(x_1, \cdots, 0)\nabla \underline{u}_{R_*} \geq 0$ at $x_*$. However, from (6.10) and the latter relation, we get that

$$\nu \nabla \underline{u}_{R_*} = \begin{cases} \nu \cdot (x_1, \cdots, 0)\frac{1}{x_1} u'_{R_*}(x_1 - T) & \text{if } x_* \in \partial\Omega \cap \{x_1 > 0\}, \\ \\ \nu \cdot (x_1, \cdots, 0)\frac{1}{x_1} u'_{R_*}(x_1 + T) & \text{if } x_* \in \partial\Omega \cap \{x_1 < 0\}, \end{cases}$$

at $x_*$, i.e., $\nu \nabla \underline{u}_{R_*} \leq 0$ at $x_*$; a contradiction. We have therefore shown that claim (6.13) holds.

Letting $R \to \infty$ in (6.13), as before, we arrive at (6.2).

The proof of the theorem is complete. □

**Remark 6.4.** If in addition $W$ satisfies relations (4.12), and (5.1) with $\mu_+ = \mu$, then there is no need to assume that $\mu_- \leq u \leq \mu$ in the assertions of Theorems 6.1, 6.2 (recall the proof of Proposition 5.1).

**Remark 6.5.** In Theorems 6.1 and 6.2, we assumed that the obstacle is smooth for the purposes of applying Hopf's boundary point lemma. In this regard, we refer to [140] for a generalization of the latter lemma to domains with corners (see also the proof of our Proposition 11.2 below).

**Remark 6.6.** If $W$ satisfies **(a")** and $W'(t) < 0$, $t \in (\mu_-, \mu)$, then the assertions of Theorems 6.1 and 6.2 can be proven easily (recall Remark 5.1).

6.2. **A Liouville-type theorem in a convex epigraph.** Adapting the proof of Theorem 6.1, using the $n$-dimensional $u_R$, we can show the following proposition. In the special case where $\Omega$ is the half-space $\mathbb{R}^n_+$, this proposition will come in handy when dealing with a class of mixed boundary value problems in Section 9; in fact, in this special case, this proposition is contained in Theorem 10.2 below (via a reflection argument)).

**Proposition 6.1.** Assume that $W \in C^2$ satisfies condition **(a")** with $W(\mu_-) = 0$ allowed. Let $\Omega$ be an entire epigraph of the form (4.10), with $\varphi$ convex and $\|\nabla\varphi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} \leq C$, for some $\alpha \in (0, 1)$ and $C > 0$. Then $u \equiv \mu$ is the only classical solution (in $C^2(\bar{\Omega})$) to the problem

$$\begin{cases} \Delta u = W'(u) & \text{in } \Omega, \\ \\ \nu \cdot \nabla u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.14}$$

where $\nu$ denotes $\partial\Omega$'s outer unit normal, such that $\mu_- \leq u \leq \mu$ and

$$u(x', x_n) \to \mu, \ \text{uniformly in } \mathbb{R}^{n-1}, \ \text{as } x_n - \varphi(x') \to \infty. \tag{6.15}$$

*Proof.* As before, by the strong maximum principle and Hopf's boundary point lemma, we deduce that $u > \mu_-$ on $\overline{\Omega}$. In fact, we claim that

$$\inf_{x\in\overline{\Omega}} u(x) > \mu_-, \tag{6.16}$$

(this is not needed in the case where $W(\mu_-) > 0$). To show this, we will argue by contradiction (in the spirit of Lemma 3.4 in [38], see also [125] and [26, 137]), namely assume that

$$u(y'_j, y_j) \to \mu_- \text{ for some } y'_j \in \mathbb{R}^{n-1} \text{ and } y_j \geq \varphi(y'_j).$$

Note that (6.15) implies that there exists an $L > 0$ such that

$$u(x', x_n) \geq \frac{\mu_- + \mu}{2} \text{ if } x' \in \mathbb{R}^{n-1} \text{ and } x_n \geq \varphi(x') + L. \tag{6.17}$$

It follows that $\varphi(y'_j) \leq y_j \leq \varphi(y'_j) + L$ for large $j$, and, passing to a subsequence, we find that

$$y_j - \varphi(y'_j) \to Y_\infty \in [0, L]. \tag{6.18}$$

Using the Neumann boundary conditions, as in [195], we can extend $u$ to a $C^2$ function in a neighborhood of $\Omega$. Then, applying interior regularity estimates (see [142]), we infer that there exists a constant $C > 0$ such that

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C,$$

(for this, as explained in [125], it is important that $\|\varphi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}$ is finite). By Lemma 6.37 in [142], we can take $\tilde{u} \in C^{2,\alpha}(\mathbb{R}^n)$ to be a smooth extension of $u$, that is $\tilde{u} = u$ in $\Omega$, such that

$$\|\tilde{u}\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C,$$

see also [125]. Now, let

$$v_j(z) = \tilde{u}(z' + y'_j, z_n + y_j), \ z = (z', z_n) \in \mathbb{R}^n.$$

Each $v_j$ solves (6.14) in

$$\Omega_j = \left\{ z_n > \varphi_j(z') \equiv \varphi(z' + y'_j) - y_j \right\}, \tag{6.19}$$

there exists $C > 0$ such that

$$\|v_j\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C \text{ for every } j, \tag{6.20}$$

$$\mu_- \leq v_j \leq \mu, \text{ and } v_j(0,0) \to \mu_-. \tag{6.21}$$

Moreover, thanks to (6.17), we have

$$v_j\left(0, \varphi(y'_j) - y_j + L\right) = u\left(y'_j, \varphi(y'_j) + L\right) \geq \frac{\mu_- + \mu}{2}. \tag{6.22}$$

In view of (6.20), and the usual diagonal argument, passing to a subsequence, we find that

$$v_j \to v_\infty \ \text{ in } C^{2,\alpha}_{loc}\left(\mathbb{R}^n\right),$$

for some $v_\infty$ with $\|v_\infty\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C$ (this $\alpha$ is in fact the same as in (6.20), see Lemma 6.1.6 in [152]), and

$$\mu_- \leq v_\infty \leq \mu \text{ in } \mathbb{R}^n, \text{ and } v_\infty(0,0) = \mu_-, \tag{6.23}$$

(recall (6.21)). Furthermore, in view of (6.18) and (6.22), we get that

$$v_\infty(0, L - Y_\infty) \geq \frac{\mu_- + \mu}{2}, \tag{6.24}$$

which implies that $v_\infty$ is not identically equal to $\mu$ in $\Omega_\infty$. From (6.18), and (6.19), we find that

$$\varphi_j(0) \to -Y_\infty \in [-L, 0].$$

Moreover, we have

$$\|\nabla\varphi_j\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} = \|\nabla\varphi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} \leq C \quad \forall\ j.$$

From the above two relations, via Arczela-Ascoli's theorem, we conclude that, for a subsequence,

$$\text{the } \varphi_j \text{ converge in } C^2_{loc}(\mathbb{R}^{n-1}) \text{ to a function } \varphi_\infty \in C^2(\mathbb{R}^{n-1}), \tag{6.25}$$

which is also convex. We write

$$\Omega_\infty = \{x_n > \varphi_\infty(x')\}\,.$$

We have that $v_\infty$ is a solution to (6.14) in $\Omega_\infty$. Indeed, if $x = (x', x_n) \in \Omega_\infty$, then $x_n > \varphi_\infty(x')$. It follows that $x_n > \varphi_j(x')$ for large $j$, and thus

$$\Delta v_\infty(x) = \lim_{j\to\infty} \Delta v_j(x) = \lim_{j\to\infty} W'\left(v_j(x)\right) = W'\left(v_\infty(x)\right).$$

Let $\nu_j(x)$ denote the outer unit normal vector to $\partial\Omega_j$ at $x \in \partial\Omega_j$, and $\nu_\infty(x)$ the corresponding vector to $\partial\Omega_\infty$. We have, for $\nu_\infty$ and $\nu_j$ evaluated at $(x', \varphi_\infty(x'))$ and $(x', \varphi_j(x'))$ respectively, that

$$\begin{aligned}
\left|\nu_\infty \nabla v_\infty\left(x', \varphi_\infty(x')\right)\right| &\leq \left|\nu_\infty \nabla v_\infty\left(x', \varphi_\infty(x')\right) - \nu_\infty \nabla v_j\left(x', \varphi_\infty(x')\right)\right| \\
&\quad + \left|\nu_\infty \nabla v_j\left(x', \varphi_\infty(x')\right) - \nu_\infty \nabla v_j\left(x', \varphi_j(x')\right)\right| \\
&\quad + \left|\nu_\infty \nabla v_j\left(x', \varphi_j(x')\right) - \nu_j \nabla v_j\left(x', \varphi_j(x')\right)\right| + \left|\nu_j \nabla v_j\left(x', \varphi_j(x')\right)\right| \\
&\overset{(6.20)}{\leq} \sup\nolimits_{B_1(x',\varphi_\infty(x'))} \left|\nabla v_\infty - \nabla v_j\right| + C\left|\varphi_\infty(x') - \varphi_j(x')\right| \\
&\quad + C\left|\nu_\infty\left(x', \varphi_\infty(x')\right) - \nu_j\left(x', \varphi_j(x')\right)\right|,
\end{aligned}$$

where, in turn, the last term can be estimated as

$$\begin{aligned}
\left|\nu_\infty\left(x', \varphi_\infty(x')\right) - \nu_j\left(x', \varphi_j(x')\right)\right| &\leq \left|\nu_\infty\left(x', \varphi_\infty(x')\right) - \nu_\infty\left(x', \varphi_j(x')\right)\right| \\
&\quad + \left|\nu_\infty\left(x', \varphi_j(x')\right) - \nu_j\left(x', \varphi_j(x')\right)\right| \\
&\leq C\left|\varphi_\infty(x') - \varphi_j(x')\right| + \sup\nolimits_{B_1(x',\varphi_\infty(x'))} \left|\nu_\infty(y) - \nu_j(y)\right|,
\end{aligned}$$

where we used that $\|\nabla\varphi_\infty\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq C$, the fact that $\nu_\infty$ and $\nu_j$ are functions of $\nabla\varphi_\infty$ and $\nabla\varphi_j$ respectively, and (6.25). Hence, by letting $j \to \infty$, we deduce that $v_\infty$ satisfies Neumann boundary conditions on $\partial\Omega_\infty$. On the other hand, in view of (6.23), recalling that $W'(\mu_-) \leq 0$, the strong maximum principle and Hopf's boundary point lemma (applied in the equation for $v_\infty - \mu_-$ in $\Omega_\infty$) imply that $v_\infty \equiv \mu_-$; a contradiction to (6.24). Thus, relation (6.16) holds.

Let $u_R$ be as in Theorems 6.1-6.2, but with $B_R \subset \mathbb{R}^n$, $R > 0$, and

$$u_R(R) = \mu_- + \delta < \inf_{x\in\overline{\Omega}} u(x),$$

where $\delta > 0$ is also chosen so that $W'(\mu_- + \delta) \leq 0$ (the point is that $W(\mu_- + \delta) > 0$). By virtue of (6.15), there exists a large $M > \max_{|x'|\leq R} \varphi(x') + R$ such that

$$u(x) > u_R(0) \geq u_R(x - Q), \quad x \in B_R(Q), \text{ where } Q = (0, \cdots, M). \tag{6.26}$$

Now, consider the family of functions:

$$\underline{u}_{R,P}(x) = \underline{u}_R(x - P), \quad P \in \Omega, \ (R > 0 \text{ fixed but arbitrary}),$$

where $\underline{u}_R$ is defined by

$$\underline{u}_R = \begin{cases} u_R, & x \in B_R, \\ \\ \mu_- + \delta, & \text{otherwise.} \end{cases} \tag{6.27}$$

Firstly, note that $\underline{u}_{R,P}$ is a weak lower solution to the equation in (6.14) (see [35], and also recall our first proof of Theorem 1.2). Moreover, if $x \in \partial\Omega$ with $|x - P| < R$, then

$$\frac{\partial \underline{u}_{R,P}}{\partial \nu} = \nu \cdot \nabla u_R(x - P) = \nu \cdot \frac{(x - P)}{|x - P|} u_R'(|x - P|) \leq 0 \ \text{ at } x,$$

by (2.9) and the convexity of $\Omega$. Whereas, if $x \in \partial\Omega$ with $|x - P| > R$, then

$$\frac{\partial \underline{u}_{R,P}}{\partial \nu} = 0 \ \text{ at } x.$$

In view of (6.26), and the above observations, keeping $R$ fixed, starting from $P = Q$, we can slide $B_R(P)$, $P \in \Omega$, to obtain that

$$u(x) \geq u_R(0), \quad x \in \bar{\Omega}, \ R > 0,$$

(keep in mind that $u$ and $\underline{u}_{R,P}$ cannot touch on $\partial B_R(P)$). Then, letting $R \to \infty$ in the above relation, via the obvious analog of (2.3), recalling that $u \leq \mu$, we conclude that $u \equiv \mu$, as desired.

The proof of the proposition is complete. □

**Remark 6.7.** The assertion of Proposition 6.1 remains true if the uniform convergence in (6.15) is replaced by pointwise, provided that we do not allow $W(\mu_-)$ to be zero. Indeed, the pointwise convergence, the boundedness of $u$, and Arczela-Ascoli's theorem, imply that, given $R > 0$, we have

$$u \to \mu, \text{ uniformly in } B_R', \text{ as } x_n \to \infty,$$

where $B_R'$ denoted the ball of radius $R$ in $\mathbb{R}^{n-1}$ with center at the origin (see the last part of the proof of Theorem 1.1 in [144]). In fact, since $\mu - u \geq 0$, this property can also be shown by means of Harnack's inequality in the linear equation for $\mu - u$, which implies that

$$\sup_{B_R(x)} (\mu - u) \leq C(R) \inf_{B_R(x)} (\mu - u) \ \ \forall x \in \mathbb{R}^n \text{ and } R > 0 \text{ such that } B_R(x) \subset \Omega,$$

(see Lemma 2.3 in [159]). Actually, the latter approach only requires that $u(x', x_n^j) \to \mu$ for some $x' \in \mathbb{R}^{n-1}$ and a sequence $x_n^j \to \infty$ as $j \to \infty$. In fact, it follows readily from the proof of Proposition 6.1 that the latter property holds if and only if

$$\sup_{\mathbb{R}^n_+} u = \mu.$$

Consequently, relation (6.26) holds true, with the corresponding minimizer $u_R$ such that $u_R(R) = \mu_-$. Now, because $W(\mu_-) > 0$, the latter minimizer satisfies the assertions of Lemma 2.1 and Proposition 2.1.

**Remark 6.8.** If $W'(t) \leq 0$, $t \in [\mu_-, \mu]$, then the special case of Proposition 6.1, where $\Omega$ is a half-space (namely $\varphi$ is a constant), follows easily, via a reflection argument, from Proposition 2.4 in [120].

6.3. **The case of smooth, bounded, star-shaped domains.** In analogy to Theorem 6.1, we have

**Proposition 6.2.** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, with outward unit normal $\nu$, which is star-shaped with respect to some point $x_0 \in \Omega$. Let $\mu_- \leq u \leq \mu$ be a classical solution to (6.14), where $W \in C^2$ satisfies conditions **(a")** with $W(\mu_-) = 0$ allowed, and (1.15). There exist numbers $R_0, \epsilon_1 > 0$, depending only on $W$, such that if $\bar{B}_{R_0}(x_0) \subset \Omega$ and $u(x) > \mu - \epsilon_1$ on $\bar{B}_{R_0}(x_0)$ then $u \equiv \mu$.

*Proof.* The proof of this proposition is in the spirit of that of Theorem 6.1. By the strong maximum principle and Hopf's boundary point lemma (see [142]), recalling that $W'(\mu_-) \leq 0$, we deduce that
$$\min_{x \in \bar{\Omega}} u(x) > \mu_-,$$
keeping also in mind that $\bar{\Omega}$ is compact (compare with (6.16)). Again, we may assume without loss of generality that $x_0 = 0$.

Let $u_R$, $R \geq R_0$, be the radial minimizers that we used in Proposition 6.1. As before, the functions in (6.27) fashion a smooth family of weak lower solutions to (6.14) for $R \geq R_0$. Let $\epsilon_1 = \mu - u_{R_0}(0)$.

Suppose that $u$ is as stated in the proposition with the above choices of $R_0, \epsilon_1$ (and $x_0 = 0$). Clearly, we have
$$u > \underline{u}_{R_0} \quad \text{on } \bar{\Omega}.$$
Now, similarly to Proposition 3.1, we do "ballooning". As $R > R_0$ increases, there are three possibilities. The first one is that there exists some $R_* > R_0$ and an $x_* \in \Omega$ such that $\underline{u}_R < u$ on $\bar{\Omega}$ for $R \in [R_0, R_*)$ and $\underline{u}_{R_*}(x_*) = u(x_*)$. The second possibility is the same as the first but with $x_* \in \partial\Omega$. The third possibility is that $u$ and $\underline{u}_R$ never touch, namely $u > \underline{u}_R$ on $\bar{\Omega}$ for every $R \geq R_0$. We make the following observations. The first scenario cannot occur because of the strong maximum principle. In the case that the last scenario holds, letting $R \to \infty$ and recalling Lemma 2.1 (for these $u_R$'s), we infer that $u \equiv \mu$ as desired. Therefore, it remains to exclude the second scenario, namely that $\underline{u}_R$ and $u$ first touch at a point $x_* \in \partial\Omega$ when $R = R_* > R_0$; note that $0 < |x_*| < R_*$. To this end, we will argue by contradiction and assume that it holds. Note first that relation (6.10) remains unchanged (notation-wise). Analogously to (6.11), we have $x\nu_x \geq 0$, $x \in \partial\Omega$. Keeping in mind that, in the case at hand, we have
$$\underline{u}'_{R_*}(|x_*|) = u'_{R_*}(|x_*|) \overset{(2.9)}{<} 0,$$
we get a contradiction.

The proof of the proposition is complete. □

**Remark 6.9.** If $\Omega$ is bounded, smooth and *convex*, there are no non-constant stable solutions to (6.14) for *any* $W$ (see [80] and [188]).

In analogy to Theorem 6.2, we can show

**Proposition 6.3.** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, with outward unit normal $\nu$, which is directionally star-shaped with respect to some direction $x_i$, $i = 1, \cdots, n$.

Let $\mu_- \leq u \leq \mu$ be a classical solution of (6.14), where $W \in C^2$ satisfies conditions **(a")** with $W(\mu_-) = 0$ allowed, and (1.15). There exist numbers $R_2, \epsilon_2 > 0$, depending only on $W$, such that if $u(x) > \mu - \epsilon_2$ on $\bar{\Omega} \cap \{|x_i| \leq R_2\}$, then $u \equiv \mu$.

## 7. Extensions: Multiple ordered solutions

Suppose that $W : \mathbb{R} \to \mathbb{R}$ is $C^2$ and there are positive numbers

$$\mu_1 < \cdots < \mu_m, \quad m \geq 2,$$

such that

$$W(\mu_1) > \cdots > W(\mu_m), \quad W'(0) \leq 0, \quad W'(\mu_i) = 0, \quad i = 1 \cdots, m,$$

and

$$W(t) > W(\mu_i), \quad t \in [0, \mu_i), \; i = 1, \cdots, m.$$

Note that at the points $\mu_i$, the potential $W$ has either minima or saddles. Obviously, we can extend $W$ outside of $[0, \mu_i]$, to a $C^2$ potential $\tilde{W}_i$, in such a way that condition **(a')** is satisfied with $\tilde{W}_i(t) - W(\mu_i)$ in place of $W$ and $\mu_i$ in place of $\mu$, $i = 1, \cdots, m$. Next, consider any

$$\epsilon \in \left(0, \min_{i=1,\cdots,m} (\mu_i - \mu_{i-1})\right), \tag{7.1}$$

with the convention that $\mu_0 = 0$, and any

$$D_i > D_i' \;\text{ where }\; D_i' \;\text{ solve }\; \mathbf{U}_i\left(D_i'\right) = \mu_i - \epsilon, \quad i = 1, \cdots, m, \tag{7.2}$$

where

$$\mathbf{U}_i''(s) = W'\left(\mathbf{U}_i(s)\right), \; s > 0; \quad \mathbf{U}_i(0) = 0, \; \lim_{s \to \infty} \mathbf{U}_i(s) = \mu_i. \tag{7.3}$$

By means of Theorem 1.2, there exist positive numbers $R_i' > D_i$, depending only on $\epsilon$, $D_i$, $\tilde{W}_i$, $i = 1, \cdots, m$, and $n$, such that if $\Omega$ has nonempty $C^2$-boundary and contains a closed ball of radius $R_i'$ then there exists a solution $u_i$ of

$$\Delta u = \tilde{W}_i'(u), \; x \in \Omega, \quad u(x) = 0, \; x \in \partial\Omega, \tag{7.4}$$

satisfying

$$0 < u_i(x) < \mu_i, \quad x \in \Omega, \tag{7.5}$$

and

$$\mu_{i-1} < \mu_i - \epsilon < u_i(x), \quad x \in \Omega_{R_i'} + B_{(R_i' - D_i)}, \quad i = 1, \cdots, m. \tag{7.6}$$

In view of (7.5), we conclude that $u_i$ solves the original problem (1.2). Thus, given $\epsilon$ and $D_i$ as in (7.1) and (7.2) respectively, if $\Omega$ contains a closed ball of radius $R''$, where $R'' = \max_{i=1,\cdots,m} R_i'$, we find that (1.2) has at least $m$ positive solutions which satisfy (7.5)–(7.6). Moreover, keeping in mind Remark 2.17, we know that these solutions are stable.

These solutions may be chosen to be ordered, in the usual sense. In other words, given $\epsilon$ and $D_i$ as in (7.1) and (7.2) respectively, there are at least $m$ positive, stable solutions of (1.2) such that

$$u_1(x) < \cdots < u_m(x), \quad x \in \Omega, \quad 1 \leq i \leq m, \tag{7.7}$$

and (7.5)–(7.6) hold (we have chosen to keep the same notation). Indeed, the solution $u_i$ can be captured by using the constant function $\mu_i$ as an upper solution; and the function $\max\{u_{i-1}(x), \underline{u}_P^i\}$ as lower solution, where $\underline{u}_P^i$ is the lower solution in (2.63) but with $\tilde{W}_i(t) - W(\mu_i)$ in place of $W(t)$, $i = 1, \cdots, m$, and $u_0 \equiv 0$. (We use again Proposition 1 in [35], see also Proposition 1 in [173], to make sure that it is a lower solution). As in the first proof of Theorem 1.2, we can sweep with the family of lower solutions $\underline{u}_Q^i$, $Q \in \Omega_{R_i'}$ to extend the lower bound on $u_i$ (due to (2.3)) from $B_{(R_i'-D_i)}(P)$ to $\Omega_{R_i'} + B_{(R_i'-D_i)}$. Moreover, the strong inequalities in (7.7) follow from the strong maximum principle. Naturally, the obtained solutions are stable (recall Remark 2.17).

We have just proven the following:

**Theorem 7.1.** Suppose that $\Omega$ and $W$ are as described in this section. Let $\epsilon$ and $D_i$ be as in (7.1) and (7.2) respectively. There exist positive constants $R_i' > D_i$, $i = 1, \cdots, m$, depending only on $\epsilon$, $D_i$, $W$ and $n$, such that if $\Omega$ contains a closed ball of radius $R'' = \max_{i=1,\cdots,m} R_i'$, then problem (1.2) has at least $m$ stable solutions $u_i$, ordered as in (7.7), such that (7.5)–(7.6) hold true.

On the other hand, assuming that $\Omega$ is bounded and smooth (a $C^3$ boundary suffices), the theory of monotone dynamical systems (see Theorem 4.4 in [188]) guarantees the existence of at least $m-1$ unstable solutions $\hat{u}_i$, $i = 1, \cdots, m-1$, of (1.2) such that

$$u_i(x) < \hat{u}_i(x) < u_{i+1}(x), \quad x \in \Omega, \quad i = 1, \cdots, m-1. \tag{7.8}$$

This can also be shown by the well known mountain pass theorem, see [100].

In summary, we have

**Theorem 7.2.** Suppose that, in addition to the hypotheses of Theorem 7.1, the domain $\Omega$ is assumed to be smooth and bounded. Then, besides of the $m$ stable solutions $u_i$ that are provided by Theorem 7.1, there exist at least $m-1$ unstable solutions $\hat{u}_i$ of (1.2), ordered as in (7.8) (keep in mind (7.7)).

The above theorem extends an old result of P. Hess [153], in the context of nonlinear eigenvalue problems (which are included in our setting, see below), where the additional assumption that $W'(0) < 0$ was imposed (see also [65] for an earlier result in the case $n = 1$). In the same context, the case $W'(0) = 0$ was allowed in [100], at the expense of assuming that $W'(\mu_i) \neq 0$, $i = 1, \cdots, m$, and some geometric restrictions on the domain. All these references considered nonlinear eigenvalue problems of the form

$$\Delta u = \lambda^2 W'(u), \ x \in \mathcal{D}, \quad u(x) = 0, \ x \in \partial\mathcal{D}, \tag{7.9}$$

where $\mathcal{D}$ is a smooth bounded domain of $\mathbb{R}^n$. By stretching variables $x \mapsto \lambda^{-1}x$, assuming that $0 \in \mathcal{D}$ (this we can do without loss of generality), keeping the same notation, we are led to the equivalent problem:

$$\Delta u = W'(u), \ x \in \Omega, \quad u(x) = 0, \ x \in \partial\Omega, \tag{7.10}$$

where $\Omega \equiv \lambda\mathcal{D}$, for $\lambda > 0$, which is plainly problem (1.2). If $\lambda$ is sufficiently large, then certainly the domain $\Omega$ contains the ball $B_{R''}$, appearing in the assertion of Theorem 7.1, but not the other way around. In contrast to our approach of using upper and lower solutions, De Figueiredo in [100] obtained the corresponding stable solutions as minimizers of the associated energy functionals (with $W$ suitably modified outside of $[0, \mu_i]$, $i = 1, \cdots, m$), and a geometric condition had to be imposed on the domain in order to ensure that they are distinct for large $\lambda$. In our case, the fact that they are distinct follows at once from (7.5) and (7.6). As we have already pointed out, in [100], the unstable solutions were constructed as mountain passes (saddle points of the energy).

**Remark 7.1.** It has been proven in [91] that if $W'(t) < 0$, $t \in (0, \mu)$, $W'(0) < 0$, or $W'(0) = 0$ but $W''(0) < 0$, $W'(\mu) = 0$, and $W'' \geq 0$ near $\mu$, then (7.9), with $\mathcal{D}$ smooth and bounded, has a unique solution with values $(0, \mu)$ when $\lambda$ is large, see also [26].

**Remark 7.2.** If $\mathcal{D}$ is a bounded domain with $C^2$-boundary, it follows from the proof of Theorem 1.2 that the corresponding stable solutions of (7.9), provided by Theorem 7.1, develop a boundary layer of size $\mathcal{O}(\lambda^{-1})$, as $\lambda \to \infty$, along the boundary of $\mathcal{D}$ (see Proposition 8.1 below for more details, and compare with the proof of Theorem 1.1 in [179], as well as with Theorem 4 in [100] and Lemma 2 in [173]). Loosely speaking, this means that the stable solutions $u_i$ converge uniformly to $\mu_i$ on the domain $\mathcal{D}$ excluding the strip that is described by $\mathrm{dist}(x, \partial\mathcal{D}) \leq |\ln\lambda|^\alpha |\lambda^{-1}$, $\alpha > 0$, as $\lambda \to \infty$. It follows from (7.8) that the corresponding unstable solutions of (7.9), provided by Theorem 7.2, also develop a (local) boundary layer behavior. In fact, if $W''(\mu_i) > 0$, the fine structure of the boundary layer of the stable solution $u_i$ is determined by the unique solution of the problem (7.3), see [26] and Remark 8.4 below. On the other side, under some restrictions on $\mathcal{D}$ and $W$, unstable solutions possessing an upward sharp spike layer on top of $u_i$, located near the most centered part of the domain, have been constructed in [93], [94], and [157] (see also [97]). The fine structure of this interior spike layer is determined by the problem

$$\Delta V = W'(V + \mu_i) \ \text{ in } \mathbb{R}^n; \quad V(x) \to 0, \ |x| \to \infty.$$

## 8. On the boundary layer of global minimizers of singularly perturbed elliptic equations

In this section, assuming only **(a')**, we will prove a general result on the size of the boundary layer of solutions of (7.9), which minimize the associated energy functional, as $\lambda \to \infty$ (recall also Remark 7.2). Setting $\varepsilon = \lambda^{-1} \to 0$, gives rise to a singular perturbation problem of the form

$$\varepsilon^2 \Delta u = W'(u), \quad x \in \mathcal{D}; \quad u(x) = 0, \ x \in \partial\mathcal{D}, \tag{8.1}$$

and in this regard it might be helpful to recall Remark 2.1.

We emphasize that, in contrast to previous results in this direction, as Theorem 1.1 in [179], here the size of the boundary layer is shown to be *independent of the dimension n*. This is due to our previous improvement over Lemma 2.2 in [179] that was made in Lemma 2.1 herein (recall the discussion preceding it, and also see Remark 8.3 below). The point is that we have not assumed any nondegeneracy on $W$ at $\mu$; in the case where $W''(\mu) > 0$ or $n = 2$, the structure of the boundary layer is well understood (recall Remark 7.2 and see Remark 8.2 below). For a different possible approach to this, see Remark 8.4 below.

The main result in this section is

**Proposition 8.1.** Suppose that $\mathcal{D}$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with $C^2$-boundary, and let $W$ satisfy assumption **(a')**. Consider any $\epsilon \in (0, \mu)$ and $D > D'$, where $D'$ as in (1.11). There exists a positive constant $\lambda_*$, depending only on $\epsilon$, $D$, $\mathcal{D}$, and $W$, such that there exists a solution $u_\lambda$ of (7.9), which minimizes the associated energy functional, satisfies

$$0 < u_\lambda(x) < \mu, \quad x \in \mathcal{D}, \tag{8.2}$$

and

$$u_\lambda(x) \geq \mu - \epsilon, \ x \in \bar{\mathcal{D}}_{(D\lambda^{-1})}, \tag{8.3}$$

provided that $\lambda \geq \lambda_*$ (recall the definition (1.6), and note that $\mathcal{D}_{(D\lambda^{-1})}$ is a connected domain for large $\lambda$). (See also the comments at the end of the assertion of Lemma 2.1).

*Proof.* As in the second proof of Theorem 1.2, recalling the discussion leading to (7.10), there exists a smooth solution of (7.9), which minimizes the associated energy and satisfies (8.2), provided that $\lambda$ is sufficiently large, say $\lambda \geq \lambda_0$, depending not just on $W$ but this time also on the domain $\mathcal{D}$.

Since $\partial\Omega \in C^2$, we know that $\Omega$ satisfies the interior ball condition (see [142]). In other words, there exists a radius $r_0 > 0$ and a family of balls $B_{r_0}(q) \subseteq \mathcal{D}$, $q \in \partial\mathcal{D}_{r_0}$ (i.e. $q \in \mathcal{D}$ with $\mathrm{dist}(q, \partial\mathcal{D}) = r_0$) such that, for each such $q$, the closed ball $\bar{B}_{r_0}(q)$ touches $\partial\mathcal{D}$ at exactly one point.

Let $\epsilon \in (0, \mu)$ and $D > D'$, where $D'$ as in (1.11). It follows from Lemma 2.1 (after a simple rescaling) that there exists a $\lambda_* > 0$, depending only on $\epsilon$, $D$, $W$, and $\mathcal{D}$ (in terms of $r_0$), and a global minimizer $u_{r_0,q}$ of the associated energy to the equation of (7.9) in $W_0^{1,2}(B_{r_0}(q))$ such that

$$0 < u_{r_0,q}(x) < \mu, \ \ x \in B_{r_0}(q), \quad \text{and} \quad u_{r_0,q}(x) \geq \mu - \epsilon, \quad x \in \bar{B}_{(r_0 - D\lambda^{-1})}(q),$$

provided that $\lambda \geq \lambda_*$. (Without loss of generality, we may assume that $\lambda_* > \lambda_0$). Thanks to Lemma A.3 below, we obtain that $u_\lambda(x) \geq u_{r_0,q}(x)$, $x \in B_{r_0}(q)$. Since the center $q$ was any point on $\partial\mathcal{D}_{r_0}$, it follows that assertion (8.3) holds true for $x \in \mathcal{D}$ such that

$$D\lambda^{-1} \leq \mathrm{dist}(x, \partial\mathcal{D}) \leq 2r_0 - D\lambda^{-1}. \tag{8.4}$$

If $W'(t) < 0$, $t \in [\mu - 2\epsilon, \mu)$, then the validity of (8.3), over the entire specified domain, follows at once via the second assertion of Lemma A.2 (this is also the case when relation (2.25) holds, recall Remark 2.3). Otherwise, we proceed as follows, see also Lemma 2 in [173]: Firstly, we cover $\bar{\mathcal{D}}_{r_0}$ by a finite number of balls of radius $\frac{r_0}{2}$ with centers on $\bar{\mathcal{D}}_{r_0}$. Secondly, if necessary, we increase the value of $\lambda_*$ such that $D\lambda_*^{-1} < \frac{r_0}{2}$. Lastly, we apply Lemma A.3 to show that

$$u_\lambda(x) \geq u_{r_0,p}(x) \geq \mu - \epsilon, \quad x \in \bar{B}_{(r_0 - D\lambda^{-1})}(p) \supseteq \bar{B}_{\frac{r_0}{2}}(p),$$

for every center $p$ of the finite covering of $\bar{\mathcal{D}}_{r_0}$, if $\lambda \geq \lambda_*$. We point out that this last part could have also been obtained from the weaker relation (2.12) (with the obvious modifications). The desired estimate (8.3) now follows from the comments leading to (8.4) and the above relation.

The proof of the proposition is complete. □

**Remark 8.1.** A similar result also holds if the domain $\mathcal{D}$ is unbounded.

**Remark 8.2.** The asymptotic behavior, as $\lambda \to \infty$, of uniformly bounded from above and below (with respect to $\lambda$), stable solutions of (7.9), where $\mathcal{D} \subseteq \mathbb{R}^n$ is bounded and smooth, has been studied in [97] in dimensions $n = 2, 3$ by techniques related to the proof of De Giorgi's conjecture in low dimensions. For a related result in $\mathbb{R}^4$, see [113]. In fact, since global minimizers are stable, and since assumption **(a')** implies that $W'(0) \leq 0$, the assertions of Proposition 8.1 when $n = 2$ follow readily from Theorem 6 in [97]; this is also the case when $n = 3$, provided that the monotonicity assumption **(b)** from our introduction is imposed.

**Remark 8.3.** Let $\epsilon$, $D$, $R' > 0$ be related as in the assertion of Lemma 2.1. By means of a simple rescaling argument (see also the proof of Theorem 1.1 in [179]), Lemmas 2.1 and

A.3 yield that the solution of (7.9), described in Proposition 8.1, satisfies $\mu - u_\lambda(x) \geq \epsilon$, if $\mathrm{dist}(x, \partial\mathcal{D}) > D\lambda^{-1}$, provided that $\lambda$ is sufficiently large (depending on $\epsilon$, $W$, and $\mathcal{D}$). Note that relation (2.12) yields the same estimate but over the smaller region that is described by $\mathrm{dist}(x, \partial\mathcal{D}) > \frac{R'}{2}\lambda^{-1}$, which *depends on $n$*, see [179].

**Remark 8.4.** Let $x_0 \in \partial\mathcal{D} \in C^2$ and $\mathcal{R}$ denote the matrix in $SO(N, \mathbb{R})$ that rotates the vector $(0, \cdots, 0, 1)$ onto the inner normal to $\partial\mathcal{D}$ at $x_0$. We can extract a sequence of $\lambda \to \infty$ such that any global minimizer $u_\lambda$, provided by Proposition 8.1, satisfies

$$u_\lambda\left(x_0 + \lambda^{-1}\mathcal{R}y\right) \to U(y),$$

uniformly on compacts, as $\lambda \to \infty$, where $U$ is some nonnegative, global minimizer (in the sense of (2.72), this can be seen as in page 104 of [96]) of the following half-space problem

$$\Delta u = W'(u),\ y \in \mathbb{R}^n_+;\quad u(y) = 0,\ y \in \partial\mathbb{R}^n_+,$$

see [26], [97] for more details, where $\mathbb{R}^n_+ = \{(y_1, \cdots, y_n) \ :\ y_n > 0\}$. Furthermore, this solution is nontrivial by virtue of Remark 8.3. Hence, by the strong maximum principle, recall **(a')**, we deduce that $U$ is positive in $\mathbb{R}^n_+$. As before, combining Lemmas 2.1 and A.3, we obtain that

$$u(y) \to \mu \ \text{ as } \ y_n \to \infty,\ \text{uniformly in } (y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1},$$

(the weaker assertion (2.12) is sufficient for this). It follows from Theorem 1.4 in [39] that $U$ depends only on the $y_n$ variable and therefore coincides with $\mathbf{U}(y_n)$ that was described in (1.12). (If $W''(\mu) > 0$ then this has been shown earlier in [26], see also [42], [86] for the weaker case (1.15) and Proposition 10.5 below).

**Remark 8.5.** In [229], the author established an asymptotic expansion of $\nu\nabla u_\varepsilon(P)$, $P \in \partial\mathcal{D}$, as $\varepsilon \to 0$, where $u_\varepsilon$ solves (8.1) for a class of nonlinearities which in particular satisfy **(c)** and (1.9) (see also [86] and [132]; see also Remark 10.4 below). As usual, the vector $\nu$ denotes the unit outer normal to $\partial\mathcal{D}$ (having assumed that it is smooth and bounded). This expansion reveals that if $P_1$ is the only point which attains the minimum of the mean curvature of $\partial\mathcal{D}$, then $P_1$ is the steepest point of the boundary layer.

**Remark 8.6.** By adapting the proof of Lemma 2.3 in [179], and that of our Proposition 8.1, we can study the boundary layer of globally minimizing solutions of inhomogeneous singular perturbation problems of the form

$$\varepsilon^2\Delta u = W_u(u, x),\ x \in \mathcal{D};\quad u(x) = 0,\ x \in \partial\mathcal{D}, \tag{8.5}$$

as $\varepsilon \to 0$, for appropriate righthand side that is more general than those that were considered in [47, 48, 106, 179], see also Lemma 7.13 in [111] and Section 13.3 in [183] (roughly, we want **(a')** to hold with $a(x)$ instead of $\mu$, for every fixed $x \in \bar{\mathcal{D}}$, for a smooth positive function $a$).

## 9. The singular perturbation problem with mixed boundary value conditions

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with $C^2$-boundary. Suppose that $\partial\mathcal{D} = \Gamma_N \cup \Gamma_D$, where $\Gamma_N$ and $\Gamma_D$ are closed and nonempty. We consider the following mixed boundary value problem:

$$\begin{cases} \Delta u = \lambda^2 W'(u) & \text{in } \mathcal{D}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{9.1}$$

where $\lambda > 0$ is a large parameter, and $\nu$ is the unit outward normal to $\Gamma_N$ at $x \in \Gamma_N$. Denote

$$W^{1,2}_{0,\Gamma_D}(\mathcal{D}) = \left\{ u \in W^{1,2}(\mathcal{D}) \ : \ u = 0 \text{ on } \Gamma_D \right\}.$$

Under assumption **(a')** on $W \in C^2$, as before, the energy functional

$$I(u) = \int_{\mathcal{D}} \left[ \frac{1}{2}|\nabla u|^2 + \lambda^2 W(u) \right] dx, \quad u \in W^{1,2}_{0,\Gamma_D}(\mathcal{D}),$$

has a global minimizer $u_\lambda$ such that $0 \leq u_\lambda \leq \mu$ (do not confuse with the usual radial minimizer $u_R$). Moreover, as in the second proof of Theorem 1.2, we have that $u_\lambda$ is nontrivial for large $\lambda$. It is more or less standard that $u_\lambda$ fashions a weak solution to (9.1) (see Chapter 5 in [56]). Then, from the theory in [235], it follows that $u_\lambda$ is a classical solution.

Similarly to Theorem 1.2, exploiting Proposition 6.1, we have the following result.

**Proposition 9.1.** Assume $\mathcal{D}$, $W$, and $u_\lambda$, as above. Given $\epsilon \in (0, \mu)$, there exist positive constants $\lambda_*, M$ such that

$$u_\lambda(x) \geq \mu - \epsilon \ \text{ if } \ \text{dist}(x, \Gamma_D) \geq M\lambda^{-1} \text{ and } \lambda \geq \lambda_*. \tag{9.2}$$

*Proof.* By using Lemma A.3 below, and sliding around a radial minimizer of radius $\lambda^{-1}R$ (with $R$ fixed large, as dictated by Lemma 2.1), we infer that there exists a constant $C > 0$ such that

$$u_\lambda(x) \geq \mu - \epsilon \ \text{ if } \ \text{dist}(x, \partial\mathcal{D}) \geq C\lambda^{-1}, \tag{9.3}$$

provided that $\lambda$ is sufficiently large.

Suppose that the assertion of the proposition is false. Then, there exist $\lambda_j \to \infty$ and $x_j \in \bar{\mathcal{D}}$ such that $u_j = u_{\lambda_j}$ satisfies

$$u_j(x_j) < \mu - \epsilon \ \text{ and } \ \lambda_j \text{dist}(x_j, \Gamma_D) \to \infty. \tag{9.4}$$

By virtue of (9.3), we deduce that the numbers

$$\lambda_j \text{dist}(x_j, \partial\mathcal{D}) \text{ remain bounded as } j \to \infty. \tag{9.5}$$

We may assume that $x_j \to x_\infty \in \partial\mathcal{D}$. Take the diffeomorphism $y = \Psi(x)$ which straightens a boundary portion near $x_\infty$, as in relation (2.8) of [195]. We may assume that $\Phi = \Psi^{-1}$ is defined in an open set containing the closed ball $\bar{B}_{2\kappa}$, $\kappa > 0$, and that $y_j = \Psi(x_j) \in B^+_\kappa = B_\kappa \cap \{y_n > 0\}$ for all $\kappa > 0$. As in [195], let

$$v_j(y) = u_j\left(\Phi(y)\right) \ \text{ for } y \in \bar{B}^+_{2\kappa}.$$

By the properties of this transformation (see [195]), we know that

$$\frac{\partial v_j}{\partial y_n} = 0 \text{ on } \Psi(\Gamma_N). \tag{9.6}$$

Moreover, we define a scaled function

$$w_j(y) = v_j(y_j + \lambda_j^{-1} y) \text{ for } y \in \bar{B}_{\kappa\lambda_j}.$$

In view of (9.5), passing to a subsequence, we may assume that the $n$-th coordinate of $\lambda_j y_j$ converges to $\ell \geq 0$, while the remaining coordinates "get away" from $\Psi(\Gamma_D)$ as $j \to \infty$. To be more precise, given $R > 0$, we have that

$$\left(y_j + \lambda_j^{-1}\bar{B}_R\right) \cap \Psi(\Gamma_D) = \emptyset$$

for $j$ sufficiently large. By interior elliptic regularity estimates [142] (applied after we have reflected $v_j$ across $\Psi(\Gamma_N)$, recall (9.6) and the above relation), as in [195], passing to a subsequence, we find that

$$w_j \to w \text{ in } C^2_{loc}(\mathbb{R}^n_+),$$

where $w$ satisfies

$$\Delta w = W'(w) \ \text{ in } \ \mathbb{R}^n \cap \{y_n > -\ell\}; \ w_{y_n} = 0 \ \text{ if } y_n = -\ell,$$

see also [26], [97], [141], and [179]. Furthermore, via (9.4), we have

$$w(0) \leq \mu - \epsilon. \tag{9.7}$$

Moreover, thanks to (9.3), we have $w(y) \geq \mu - 2\epsilon$ if $y_n \geq C'$ for some $C' > 0$ (assuming that $2\epsilon < \mu$). This, in particular, implies that $v$ is nontrivial. Now, as in Remark 8.4, we see that $w \to \mu$, uniformly in $\mathbb{R}^{n-1}$, as $y_n \to \infty$. On the other hand, Proposition 6.1 implies that $w \equiv \mu$ which is in contradiction to (9.7).

The proof of the proposition is complete. □

**Remark 9.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ which is symmetric with respect to some hyperplane, say $\{x_1 = 0\}$, and $W$ as in Proposition 9.1 and *even*. Let $\mathcal{D} = \Omega \cap \{x_1 > 0\}$, $\Gamma_N = \partial\Omega \cap \{x_1 \geq 0\}$, and $\Gamma_D = \bar{\Omega} \cap \{x_1 = 0\}$. Applying Proposition 9.1, yields a positive solution to (9.1) which satisfies (9.2). (Some care is required at the junction points on $\partial\Omega \cap \{x_1 = 0\}$, but this regularity issue may be treated by an approximation argument, as described in Remark 1.4, see also [107, 143]). Reflecting this solution oddly across the plane $\{x_1 = 0\}$, we obtain a solution to the Neumann problem

$$\Delta u = \lambda^2 W'(u) \text{ in } \Omega; \ \frac{\partial u}{\partial \nu} \text{ on } \partial\Omega, \tag{9.8}$$

which converges, in $L^1(\Omega)$, to the step function

$$\mu\chi_{\Omega\cap\{x_1>0\}} - \mu\chi_{\Omega\cap\{x_1<0\}},$$

as $\lambda \to \infty$ ($\chi$ denotes the usual characteristic function).

In the general case, where $\Omega$ is not symmetric, under some non-degeneracy assumptions, this type of transition-layered solutions have been constructed in two and three dimensions, via perturbation arguments, by [156], [170], and [216] (see also the references in [203]).

If $\Omega \subset \mathbb{R}^2$ is smooth, bounded, and symmetric with respect to the coordinate axis, in the same manner, we can construct solutions to (9.8) that converge, in $L^1(\Omega)$, to the step function

$$\mu\chi_{\Omega\cap\{x_1x_2>0\}} - \mu\chi_{\Omega\cap\{x_1x_2<0\}},$$

as $\lambda \to \infty$ (see also a related open question in [131]). Analogous constructions hold in higher dimensions, recall our discussion about "saddle" solutions from the introduction.

The paper [134] contains an analog of Theorem 1.1 for problem (9.1), with $\lambda > 0$ fixed, in the case where for each $x \in \Gamma_N$ there is a $\rho > 0$ such that $\mathcal{B}_\rho(x)$ is convex, where $\mathcal{B}_\rho(x)$ denotes the connected component of $B_\rho(x) \cap \mathcal{D}$ such that $x \in \bar{\mathcal{B}}_\rho(x)$. We believe that there is also a corresponding analog of Theorem 1.2. To support this, let us sketch the proof of the following proposition.

**Proposition 9.2.** Assume that $W \in C^2$ satisfies **(a)**, $\lambda > 0$, and $\mathcal{D}$ as in this section with $\Gamma_N$ convex in the above sense. Given $\epsilon \in (0, \mu)$, there exist $R_*, C > 0$ such that the existence of $x_* \in \Omega$ such that $\mathcal{B}_{R_*}(x_*) \cap \Gamma_D = \emptyset$ implies that problem (9.1) has a positive solution $u < \mu$ verifying (1.3). Moreover, there exists a $C > 0$ such that

$$u(x) \geq \mu - \epsilon, \quad \text{if } \mathcal{G}(x, \Gamma_D) \geq C,$$

here $\mathcal{G}(x, \Gamma_D)$ denotes the geodesic distance of $x$ from $\Gamma_D$, namely

$$\mathcal{G}(x, \Gamma_D) = \inf_{\gamma(x,\Gamma_D)} \mathcal{H}^1\left(\gamma(x, \Gamma_D)\right),$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure and the infimum is taken on the set of the absolutely continuous paths $\gamma(x, \Gamma_D) \subset \bar{\mathcal{D}}$ joining $x$ to $\Gamma_D$.

*Proof.* (Sketch) Plainly note that the function $\underline{u} = u_{R_*}(x - x_*)$, $x \in \mathcal{B}_{R_*}(x_*)$, zero otherwise, is a lower solution to (9.1). The key point is that the convexity property of $\Gamma_N$ implies that

$$\frac{\partial \underline{u}}{\partial \nu} \leq 0 \quad \text{on } \Gamma_N.$$

Then, we can slide around that lower solution in $\mathcal{D}$, as long as we stay away from $\Gamma_D$ (in the geodesic sense), to obtain the desired lower bound. □

## 10. Some one-dimensional symmetry properties of certain solutions to the Allen-Cahn equation

10.1. **Symmetry of entire solutions.** Many authors have studied the one-dimensional symmetry of certain entire solutions to problem (1.22) with $W$ as in (1.23), namely

$$\Delta u + u(1 - u^2) = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{10.1}$$

Their study was motivated by *De Giorgi's conjecture* (recall Remark 5.2) and *Gibbons' conjecture*. The latter claims that any solution to (10.1) which tends to $\pm 1$ as $x_1 \to \pm\infty$, *uniformly* in $\mathbb{R}^{n-1}$, is one-dimensional, i.e.

$$u(x) = \tanh\left(\frac{x_1 - a}{\sqrt{2}}\right) \quad \text{for some } a \in \mathbb{R}. \tag{10.2}$$

As we have already pointed out, the former conjecture was motivated from the theory of minimal surfaces. On the other hand, the latter conjecture was motivated from a problem in cosmology theory (see [139]).

**Remark 10.1.** Keep in mind that problem (10.1) is invariant under translations and rotations.

Gibbons' conjecture was proven almost at the same time by three different approaches: in [31] by probabilistic arguments, in [42] by the sliding method, and in [118] based on [38]. In fact, it was proven earlier for dimensions up to three in [137]; see also [138] for a different proof which holds up to dimension five. The conjecture of Gibbons' also follows from a stronger result that can be found in [31, 76] (see also [73]). The latter says that if $u$ solves (10.1), such that it possesses an unstable level set (say $u = 0$) which is a globally Lipschitz graph, then $u$ is as in (10.2) (possibly after a rotation and translation).

In this section, we will present some related new one-dimensional symmetry results, based on Proposition 3.1 as well as on an old result in [74] which does not seem to have been exploited up to this moment. In particular, we are able to provide a new proof of the well known Gibbons' conjecture.

After this section was written, we found that the same result of Theorem 10.1 below, *under the additional assumption that $W'$ is odd*, was proven previously in [109] (see also [253] for a generalization to the quasi-linear setting, where the oddness assumption is not stated explicitly in the statement of Theorem 1.3 therein but used in the proof). The strategy in the latter references was to take advantage of the oddness of $W'$, adapting some techniques from [38] and [42], to show that the solution under consideration is odd (in a certain direction); this property then reduces the one-dimensional symmetry problem to Gibbons' conjecture which was already resolved (recall our previous discussion). Moreover, it was assumed in [109] that (1.15) holds (in this regard, see Remark 10.2 below). On the other side, the proof in [109] holds for $W'$ Lipschitz (see, however, Remark 10.2 below).

Our main result is

**Theorem 10.1.** Let $u \in C^2(\mathbb{R}^n)$ be a solution to (10.1) such that there exists a point $P$ on the hyperplane $\{x_1 = 0\}$, say the origin, such that

$$u(P) = 0 \text{ and } u > 0 \text{ in } \mathbb{R}^n \cap \{x_1 < 0\}, \tag{10.3}$$

then $u$ is one-dimensional of the form (10.2).

*Proof.* As we showed in Remark 5.2, we have $|u(x)| < 1$, $x \in \mathbb{R}^n$. Similarly to the proof of Proposition 3.1 (see also Remark 3.1), we have

$$u_R(x - Q) < u(x), \quad x \in B_R(Q), \tag{10.4}$$

provided that $B_R(Q) \subset \mathbb{R}^n \cap \{x_1 < 0\}$, where $u_R$ is a solution to problem (2.6) such that (2.2) holds. Since $u$ is positive in $\mathbb{R}^n \cap \{x_1 < 0\}$, we can slide the ball $B_R(Q)$ (keeping $R$ fixed) so that it is tangent to the hyperplane $\{x_1 = 0\}$ at the origin, while keeping (10.4). In other words, relation (10.4) holds, with $Q = (-R, 0, \cdots, 0)$, for all $R > 0$. In particular, we have

$$u(x_1, 0, \cdots, 0) > u_R(R - x_1), \quad x_1 \in (-R, 0), \text{ (with the obvious notation)}, \tag{10.5}$$

and $u(0, \cdots, 0) = u_R(R) = 0$, for all $R > 0$. By Hopf's boundary point lemma (in the equation for $u - u_R$), we deduce that

$$u_{x_1}(0, \cdots, 0) < u_R'(R) \text{ for all } R > 0,$$

(clearly $u$ cannot be identically equal to $u_R(x - Q)$ in $B_R(Q)$). So, recalling that $u_R'(R) < 0$, we arrive at

$$[u_{x_1}(\mathbf{0})]^2 > [u_R'(R)]^2 \text{ for all } R > 0,$$

where $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$. Note that the left-hand side of the above relation does not depend on $R$. Now, letting $R \to \infty$ (do ballooning), and recalling Lemma 2.2 (see also (2.13)), we infer that

$$[u_{x_1}(\mathbf{0})]^2 \geq 2W(0).$$

On the other hand, it is known that every bounded solution to (10.1) satisfies the gradient bound (2.57), and the only solutions for which equality is achieved at some point are one-dimensional of the form (10.2) (see Theorem 5.1 in [74]). The above relation clearly implies that equality is achieved at $x = \mathbf{0}$ (recall that $u(\mathbf{0}) = 0$). Consequently, the solution $u$ is one-dimensional.

The proof of the theorem is complete. $\square$

It is well known that there is a deep connection between the "blown-down" level sets of solutions to (10.1) and the theory of minimal surfaces, see for example [3], [102], [203] and [218]. Let us suggest a naive argument which connects Theorem 10.1 to the theory of minimal surfaces. If $u$ satisfies the assumptions of Theorem 10.1, its zero set near the origin is a graph of $x_1$ over $\mathbb{R}^{n-1}$ ($u_{x_1}(\mathbf{0}) < 0$, thanks to Hopf's lemma, so we can apply the implicit function theorem) which is tangent to the plane $\{x_1 = 0\}$. In the "blown-down" problem (assuming for the sake of our argument that $u$ is a minimizer in the sense of [159]), near the origin, we get two minimal graphs (the one being a plane) which are tangent at the origin and the one is above the other. The strong maximum principle for minimal surfaces, see [88] and Lemma 1 in [221], tells us that both surfaces are planes. This property can be rephrased as saying that, as we translate a hyperplane towards a minimal surface, the first point of contact must be on the boundary.

**Remark 10.2.** The assertion of Theorem 10.1 remains true for solutions $u$ of (1.22), with $W \in C^3$ as in Proposition 3.1, provided that we assume in advance that $\mu_- < u < \mu$ and $W(t) \geq 0$, $t \in [\mu_-, \mu]$ ($\mu_- < 0 < \mu$). In fact, we can easily relax the regularity of $W$ to $W'$ being Lipschitz continuous, assuming that $-W'(t) \geq ct$ for $t \geq 0$ near 0 (instead of $W''(0) < 0$, recall Remark 3.3). To this end, we have to use Lemmas 3.2-3.3 in [38] instead of our Proposition 3.1, and the recent result in [125] which says that the gradient bound (2.57) is still valid for $W'$ merely Lipschitz continuous.

The assertion of Theorem 10.1 also remains true if $W'$ is Lipschitz continuous, $W(t) \geq 0$ for $t \in [\mu_-, \mu]$, $W(t) > 0$ for $t \in [0, \mu)$, $W(\mu) = 0$, and $W'(\mu) = 0$, provided that we additionally assume that $\mu_- \leq u \leq \mu$ in $\mathbb{R}^n$ and $u(x_1, x') \to \mu$ as $x_1 \to -\infty$ for some $x' \in \mathbb{R}^{n-1}$. Indeed, thanks to Remark 6.7, the last assumption implies uniform convergence over compacts of $\mathbb{R}^{n-1}$. We note that, in the case where $W'(0) > 0$, we have to use $u_R$ as in Lemma 2.3 (we can perform the sliding method even if $\max\{u_R, 0\}$ is not a weak lower solution).

**Remark 10.3.** It has been shown recently in [126] that any *energy minimizing* solution (as described in [159]) to (10.1) is one-dimensional provided that it is positive for $x_1 < 0$ (it is not required a-priori that the level set of $u$ touches $x_1 = 0$ at some point).

**Remark 10.4.** The ballooning and sliding arguments of Theorem 10.1, together with the gradient bound (2.57), can give a different proof of relation (4.5) in [225], namely that any saddle solution of (1.22) satisfies $u_{x_1}(0, x_2) \to \sqrt{2W(0)}$ as $x_2 \to \infty$ (see also [70], [99]). In fact, we can show this without assuming (1.15). For $W$'s enjoying the qualitative properties of (1.23), the rate of this convergence is exponentially fast (see [99]). In higher dimensions, it has been remarked in [72] that this convergence is of algebraic rate. Combining our approach with the fact that, for such $W$'s, there holds

$$u_R'(R) = -\sqrt{2W(0)} + \frac{(N-1)\int_0^\mu \sqrt{2\left(W(0) - W(t)\right)}dt}{\sqrt{2W(0)}} R^{-1} + \mathcal{O}(R^{-2}) \text{ as } R \to \infty,$$

see [229], we may quantify this rate. For example, for the three-dimensional saddle solution in [5], we get

$$u_{x_1}(0, x_2, x_3) = -\sqrt{2W(0)} + \mathcal{O}\left(\frac{1}{\sqrt{x_2^2 + x_3^2}}\right),$$

and

$$u_{x_2}^2(0, x_2, x_3) + u_{x_3}^2(0, x_2, x_3) \leq \mathcal{O}\left(\frac{1}{\sqrt{x_2^2 + x_3^2}}\right) \quad \text{as } x_2^2 + x_3^2 \to \infty.$$

Similarly to Theorem 10.1, we can show

**Proposition 10.1.** Assume that $W \in C^2$ satisfies condition **(a')** and is even. Let $-\mu \leq u \leq \mu$ be a solution to (1.22) such that

$$\sup_{\mathbb{R}^n_-} u = \mu, \quad \text{where } \mathbb{R}^n_- = \mathbb{R}^n \cap \{x_1 < 0\},$$

and is periodic in the remaining variables $(x_2, \cdots, x_n)$, namely $u(x_1, x_2, \cdots, x_n) \equiv u(x_1, x_2 + T_2, \cdots, x_n + T_n)$ for some $T_i \in \mathbb{R}$. Then, $u$ is one-dimensional in $x_1$ and non-increasing.

*Proof.* Firstly, by the strong maximum principle, unless $u \equiv \mu$, the periodicity of $u$ implies that

$$\sup_{|x_1| \leq L} u = \max_{|x_1| \leq L} u = c_L < \mu \quad \text{for every } L > 0.$$

It follows that there exists a sequence of points $A_j = (a_j, a_j') \in \mathbb{R}^n_-$ such that $a_j \to -\infty$, and $u(A_j) \to \mu$. As we discussed in Remark 6.7, via Harnack's inequality, we have that

$$u \to \mu, \text{ uniformly on compact subsets of } \mathbb{R}^{n-1}, \text{ as } x_1 \to -\infty.$$

By the periodicity of $u$ in the remaining variables $(x_2, \cdots, x_n)$, we find that

$$u \to \mu, \text{ uniformly in } \mathbb{R}^{n-1}, \text{ as } x_1 \to -\infty. \tag{10.6}$$

Given $R > 0$, let $u_R$ be a minimizer as provided by Lemma 2.1, extended by zero outside of $B_R$. By virtue of (10.6), we can center the ball $B_R$ at a point $Q \in \mathbb{R}^n_-$ so that

$$u(x) > u_R(0) \geq u_R(x - Q) \text{ in } B_R(Q),$$

say $Q = (-Q_1, 0, \cdots, 0)$ with $Q_1 > R$ sufficiently large.

If $u > 0$ in $\mathbb{R}^n$, we can slide the ball $B_R(Q)$ around in $\mathbb{R}^n$ to get that

$$u(x) \geq u_R(x - Q) \quad \forall\, x, Q \in \mathbb{R}^n.$$

Taking $Q = x$, yields that $u(x) \geq u_R(0)$, $x \in \mathbb{R}^n$. Since $R > 0$ was arbitrary, in view of (2.3), and recalling that $u \leq \mu$, we conclude that $u \equiv \mu$.

Otherwise, $u$ has to vanish somewhere. By virtue of (10.6), and the periodicity of $u$ in the variables $(x_2, \cdots, x_n)$, we may assume that (10.3) holds for some $P$ on the hyperplane $\{x_1 = 0\}$. Sliding the ball $B_R(Q)$ in $x_1 < 0$, until it is tangent at $P$, to obtain a similar relation to (10.5), and recalling (2.13), we can conclude as in Theorem 10.1 that $u$ depends only on $x_1$ and is monotone.

The proof of the proposition is complete. ∎

Moreover, we can show:

**Proposition 10.2.** Assume that $W$ is as in Remark 10.2 and $W'(t) > 0$, $t \in (\mu_-, 0)$. Then, there does not exist a solution $u \in C^2(\mathbb{R}^n)$ to (1.22) such that $\mu_- \le u \le \mu$ and the level set $\{x \in \mathbb{R}^n : u(x) = 0\}$ is bounded.

In the case where the assumption $W \ge 0$ is violated, say when $W(\mu_-) < 0 = W(\mu)$, then $u$ has to be radially symmetric with respect to some point $x_0 \in \mathbb{R}^n$ and increasing (see Theorem 3.3 in [118]).

10.2. **One-dimensional symmetry in half-spaces.** Consider the problem

$$\begin{cases} \Delta u = W'(u) & \text{in } \mathbb{R}^n_- = \{x_1 < 0\}, \\ u = 0 & \text{on } \{x_1 = 0\}, \\ u > 0 & \text{in } \mathbb{R}^n_-. \end{cases} \tag{10.7}$$

As we have already seen in Remark 8.4, this type of problems arise after blowing-up, close to the boundary, singular perturbation problems of the form (8.1) (see also Proposition 9.1).

The following result was proven by Angenent in [26] by the method of moving planes:

**Proposition 10.3.** Assume that $W \in C^2$ satisfies $W'(0) = 0$, $W''(0) < 0$, (1.15), $W'(\mu) = 0$, $W''(\mu) > 0$, and $W'(t) > 0$, $t > \mu$. Then, any bounded solution to (10.7) depends only on the $x_1$ variable (such solution exists and is strictly decreasing in $x_1$, recall (1.12)).

In [38], the authors relaxed the condition $W''(\mu) > 0$ to $W'$ being non-decreasing near $\mu$ and allowed for $W'$ merely Lipschitz (and also included the case $W'(0) < 0$). In fact, the condition $W'$ being non-decreasing near $\mu$ is not needed, as shown in [120] (see also [110] for a different approach). The behavior of $W$ near $t = 0$ has been relaxed in [120]. As a matter of fact, there is no need to assume something for the behavior of $W$ near $t = 0$, provided that $n \le 5$ (see [124], and the references therein, where $W' \in C^1$ is also required for $n = 4, 5$). One of the main results that was used in the aforementioned references is Theorem 1 of [37] (see also Theorem 1.4 in [39]), which says that if $u$ is a bounded solution to (10.7), where $W'$ is Lipschitz continuous, with $M = \sup_{\mathbb{R}^n_-} u$, then $u$ is one-dimensional and monotone provided that $W'(M) \ge 0$ (furthermore, $W'(M) = 0$); see also Proposition 10.5 as well as Remarks 10.5 and 10.6 herein).

Based on Theorem 10.1, we can provide a *completely different* proof of Proposition 10.3, while also removing the condition $W''(\mu) > 0$. The drawback of our approach is that we impose a higher degree of regularity on $W$, in order to apply Proposition 3.1 which is based on bifurcation arguments.

**Proposition 10.4.** Assume that $W \in C^3$ satisfies the hypotheses in Proposition 10.3, except from $W''(\mu) > 0$. Then, the same assertion of the latter proposition holds.

*Proof.* Firstly, note that, as in Proposition 5.1, it follows that $0 < u < \mu$ in $\mathbb{R}^n_-$ (see also Lemma 2.4 in [124]). Then, arguing as in Theorem 10.1, we can show that

$$u_{x_1}^2 \ge 2W(0) \quad \text{on } \{x_1 = 0\}. \tag{10.8}$$

Now, let

$$\tilde{W}(t) = \begin{cases} W(t), & t \ge 0, \\ W(-t), & t < 0, \end{cases} \quad \text{and} \quad \tilde{u}(x) = \begin{cases} u(x_1, \cdots, x_n), & x_1 \le 0, \\ -u(-x_1, \cdots, x_n), & x_1 > 0. \end{cases}$$

Since $W'(0) = 0$, it follows that $\tilde{W} \in C^2$. Clearly $\tilde{u} \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \backslash \{x_1 = 0\})$, and satisfies

$$\Delta \tilde{u} = \tilde{W}'(\tilde{u}) \quad \text{in } \mathbb{R}^n \backslash \{x_1 = 0\}.$$

In particular, since the righthand side belongs in $C^\alpha(\mathbb{R}^n)$, $\alpha \in (0, 1)$, standard interior Schauder estimates (see [142]) tell us that $\tilde{u} \in C^{2+\alpha}(\mathbb{R}^n)$. Hence, we infer that $\tilde{u}$ is a classical bounded solution to the above equation. Moreover, by its construction $\tilde{u}$ is odd with respect to $x_1$. It follows, via (10.8), that

$$|\nabla \tilde{u}|^2 \geq 2W(\tilde{u}) \quad \text{on } \{x_1 = 0\}.$$

Since $\tilde{W}(t) \geq 0$, $t \in \mathbb{R}$, $\tilde{W} \in C^2$, and vanishes at $t = \pm\mu$, Theorem 5.1 in [74] yields that $\tilde{u}$ is one-dimensional. The assertion of the proposition follows immediately.

The proof of the proposition is complete. □

Similarly, arguing as in the proof of Proposition 10.1, we can show the following proposition.

**Proposition 10.5.** Suppose that $W \in C^2$ satisfies **(a')** and (1.15). If $u$ is a solution to (10.7) such that

$$\sup_{\mathbb{R}^n_-} u = \mu, \tag{10.9}$$

then $u$ is one-dimensional.

**Remark 10.5.** Noting that most assertions of Lemma 2.1 continue to hold for $W \in C^{1,1}$, it is not hard to see that Proposition 10.5 holds for $W$ with this regularity. In this regard, note that the Lipschitz continuity of $W'$ allows for the strong maximum principle to be applied in the linear equation for $u - u_R$. A point to be stressed is that the gradient bound (2.57), proven in [74] under the assumption $W \in C^2$, was recently generalized, allowing for $W$ to be $C^{1,1}$, in [125]. Moreover, in the case where $W'$ is merely Lipschitz continuous, some care is needed when reflecting $u$ (see Corollary 1.3 in [122]).

The above proposition was proven originally under the stronger assumption (10.6) by Clément and Sweers in [86], see Proposition 2.5 therein. They also assumed that $W \in C^{2+\alpha}$, $\alpha \in (0, 1)$. Their approach was based on comparison arguments with suitable one-dimensional upper and lower solutions and shooting arguments. Subsequently, it was extended to more general equations in [37], by means of the sliding method (see also Theorem 1.4 in [39] and Theorem 4.7 in [194]), assuming merely that (10.9) holds and that $W'$ is Lipschitz continuous.

So far, the arguments in this subsection have been based in reflecting $u$ oddly across $\{x_1 = 0\}$, which is possible since $W'(0) = 0$. If $W'(0) < 0$, this ceases to be an option. Nevertheless, taking advantage of a recent result of [124] which extends the gradient bound (2.57) to the case of half-spaces, under the additional assumptions $W'(0) \leq 0$ and $u \geq 0$, we can show:

**Proposition 10.6.** The assertions of Propositions 10.4, 10.5 remain true if $W'(0) < 0$ and (1.9) hold, together with $W'(t) > 0$, $t > \mu$, for Proposition 10.4; relation (1.15) for Proposition 10.5.

*Proof.* As in Proposition 10.4 (recalling that Proposition 3.1 works for such $W$), we find that (10.8) holds. On the other side, by Theorem 1.4 in [124], we have

$$|\nabla u|^2 \leq 2W(u) \quad \text{in } \mathbb{R}^{n-1}_-. \tag{10.10}$$

In particular, from (10.8) and the above relation with $x_1 = 0$ (recalling that $u = 0$ there), we obtain that

$$u_{x_1} = -\sqrt{2W(0)}, \ u_{x_i} = 0, \ i = 2, \cdots, n, \ \text{on } \{x_1 = 0\}. \tag{10.11}$$

However, it has *not* been shown in [124] that, if equality in (10.10) is achieved at some point on $\{x_1 \leq 0\}$, the solution is one-dimensional (actually, this was shown in the subsequent paper [125] in the case where equality is achieved at an interior point which in addition is noncritical for $u$). Rather than proceeding in this direction, we will argue as follows. From the relation that corresponds to (10.5), recalling (2.18) and (2.19), we get

$$u(x) \geq \mathbf{U}(-x_1) \ \text{ in } \mathbb{R}^{n-1}_{-}, \tag{10.12}$$

where $\mathbf{U}$ is as in (1.12) (in the analog of (10.4), when the ball is tangent at $(0, x')$, we consider any strip $[-L, 0] \times \mathbb{R}^{n-1}$, look only at the $x'$-slice, and let $R \to \infty$). On the other hand, since both $u(x)$ and $\mathbf{U}(-x_1)$ solve (10.7), by the strong maximum principle and Hopf's boundary point lemma (applied to the linear equation for $u - \mathbf{U}$), we deduce that either $u_{x_1} < -\mathbf{U}'(0) = -\sqrt{2W(0)}$ on $\{x_1 = 0\}$ or $u(x) \equiv \mathbf{U}(-x_1)$ in $\mathbb{R}^{n-1}_{-}$. In view of (10.11), we conclude that the latter scenario holds.

The proof of the proposition is complete. □

As noted in [39], one-dimensional symmetry results for (10.7) can be thought of as extensions of the Gidas, Ni and Nirenberg [140] symmetry result for spheres, when the radius of the sphere increases to infinity while a point on the boundary is being kept fixed. This is essentially what we do in Theorem 10.1. This procedure that we apply can be appropriately named "method of expanding spheres".

**Remark 10.6.** After this subsection was written, we found the paper [158], where it was shown that solutions of (10.7), having values in $(0, \mu)$, with $W'$ Lipschitz and $W(t) > W(\mu) = 0$ for $t \in [0, \mu)$, satisfy the gradient bound (10.10) on the hyperplane $\{x_1 = 0\}$, which is enough for our purposes. Actually, the approach of [158], for establishing the one-dimensional symmetry of solutions to (10.7), is similar in spirit to ours.

Armed with this information, it is not hard to see that the assertion of Proposition 10.5 continues to hold solely under the aforementioned assumptions (if $W'(0) > 0$, we have to use $u_R$ as in Lemma 2.3). Hence, we can essentially recover the general result of [37].

10.3. **A rigidity result.** The following rigidity result was proven in Theorem 4.2 of [125], assuming additionally that $W'(t) \leq 0$, $W'(t) + W'(-t) \leq 0$, $t \in (0, \mu)$, and that $W'$ is non-decreasing near $\mu$. The second condition is clearly satisfied if $W'$ is odd. In this regard, it might be useful to recall our discussion preceding Theorem 10.1, related to [109] where the authors also assumed additionally that $W'$ is odd. This is not a coincidence, since both [109] and [125] employ modifications of the method of moving planes. In contrast, making use of the arguments from Theorem 10.1, we will see that the aforementioned assumptions on $W$ are not needed.

**Theorem 10.2.** Suppose that $W \in C^{1,1}(\mathbb{R})$, $W(t) \geq 0$ for $t \in \mathbb{R}$, $W(\mu) = 0$, and $W(t) > 0$ for $t \in (\mu_-, \mu)$. If $u \in C^2(\mathbb{R}^n)$ satisfies (1.22), $\mu_- \leq u \leq \mu$ in $\mathbb{R}^n$, and

$$u \to \mu, \ \text{uniformly in } \mathbb{R}^{n-1}, \ \text{as } x_1 \to \pm\infty, \tag{10.13}$$

then

$$u \equiv \mu.$$

*Proof.* Since
$$W(t)=\int_{\mu}^{t}W'(s)ds,$$
and $W(t)>0$, $t\in(\mu_-,\mu)$, there exists a sequence
$$\epsilon_j\to 0^+ \text{ such that } W'(\mu-\epsilon_j)\leq 0.$$
For every $j\gg 1$, we intend to show that
$$u\geq \mu-\epsilon_j \ \text{ in } \mathbb{R}^n,$$
which clearly implies the assertion of the proposition. To this end, we argue by contradiction, namely assume that
$$u(x_0)<\mu-\epsilon_j \text{ for some } x_0\in\mathbb{R}^n. \tag{10.14}$$
In the sequel, for notational convenience, we will drop the subscript $j$ from $\epsilon$. By virtue of (10.17), we can define
$$\ell_-=\sup\{s\in\mathbb{R}:\ u(x_1,x')>\mu-\epsilon \text{ if } x_1<s \text{ and } x'\in\mathbb{R}^{n-1}\},$$
and
$$\ell_+=\inf\{s\in\mathbb{R}:\ u(x_1,x')>\mu-\epsilon \text{ if } x_1>s \text{ and } x'\in\mathbb{R}^{n-1}\}.$$
In view of (10.14), we get that $\ell_\pm\in\mathbb{R}$ and $\ell_-<\ell_+$.

Let $u_R$ be an energy minimizing solution to the following problem:
$$\Delta u_R=W'(u_R),\ \mu-\epsilon<u_R<\mu \text{ in } B_R;\ u_R=\mu-\epsilon \text{ on } \partial B_R,$$
as provided by Lemma 2.1. In fact, it is easy to see that most assertions of Lemma 2.1 continue to hold if $W\in C^{1,1}$ instead of $C^2$, and $u_R=m$ on $\partial B_R$ with $m\in(0,\mu)$ such that $W'(m)\leq 0$ (with the obvious modifications). By the uniform asymptotic behavior of $u$ as $x_1\to-\infty$, we deduce that, given $R>0$, there exists $Q_1>R-\ell_-$ such that
$$u(x)>u_R(0)\geq u_R(x-Q),\ x\in B_R(Q),\ \text{where } Q=(-Q_1,0,\cdots,0).$$
Since $u>\mu-\epsilon$ if $x_1<\ell_-$, and $W'(\mu-\epsilon)\leq 0$, we can slide the ball $B_R(Q)$ around in $\{x_1<\ell_-\}\times\mathbb{R}^{n-1}$ (as usual), to arrive at
$$u(x_1,x')>u_R(x_1+R-\ell_-),\ \ x_1\in(\ell_--R,\ell_-),\ x'\in\mathbb{R}^{n-1},$$
using the notation $u_R(|x|)=u_R(x)$, $x\in B_R$ (with the obvious meaning). Making use of the obvious analog of (2.19) (loosely speaking, letting $R\to\infty$ in the above relation), we obtain that
$$u(x_1,x')\geq U(\ell_--x_1),\ \ x_1\leq\ell_-,\ x'\in\mathbb{R}^{n-1},$$
where here $U\in C^2[0,\infty)$ denotes the unique classical solution to
$$U''=W'(U),\ s>0;\ U(0)=\mu-\epsilon,\ \lim_{s\to\infty}U(s)=\mu,$$
we note that $U'>0$. Similarly, we have
$$u(x_1,x')\geq U(x_1-\ell_+),\ \ x_1\geq\ell_+,\ x'\in\mathbb{R}^{n-1}.$$

On the other side, from the definition of $\ell_-$, there exist sequences $(x_1)_j\geq\ell_-$ and $(x')_j\in\mathbb{R}^{n-1}$ such that
$$(x_1)_j\to\ell_- \text{ and } u\left((x_1)_j,(x')_j\right)\leq\mu-\epsilon. \tag{10.15}$$
Let
$$v_j(x_1,x')=u\left(x_1,x'+(x')_j\right),\ \ x_1\in\mathbb{R},\ x'\in\mathbb{R}^{n-1}.$$

Each $v_j$ satisfies (1.22), $\mu_- \le v_j \le \mu$,

$$v_j\left((x_1)_j, 0\right) \le \mu - \epsilon, \text{ where } (x_1)_j \to \ell_-,$$

$v_j(x_1, x') \ge U(\ell_- - x_1)$ if $x_1 \le \ell_-$, $x' \in \mathbb{R}^{n-1}$; $v_j(x_1, x') \ge U(x_1 - \ell_+)$ if $x_1 \ge \ell_+$, $x' \in \mathbb{R}^{n-1}$. Making use of standard elliptic regularity estimates [142], and the usual diagonal-compactness argument, passing to a subsequence, we find that $v_j \to v_\infty$ in $C^2_{loc}(\mathbb{R}^n)$, where $v_\infty$ satisfies (1.22), $\mu_- \le v_\infty \le \mu$, $v_\infty(\ell_-, 0) \le \mu - \epsilon$, and

$$v_\infty(x_1, x') \ge U(\ell_- - x_1) \text{ if } x_1 \le \ell_-,\ x' \in \mathbb{R}^{n-1};\ v_\infty(x_1, x') \ge U(x_1 - \ell_+) \text{ if } x_1 \ge \ell_+,\ x' \in \mathbb{R}^{n-1}. \tag{10.16}$$

It follows that $v_\infty(\ell_-, 0) = \mu - \epsilon$ and

$$\partial_{x_1} v_\infty(\ell_-, 0) \le -U'(0) = -\sqrt{2W(\mu - \epsilon)} = -\sqrt{2W(v_\infty(\ell_-, 0))}.$$

Similarly to the proof of Proposition 10.6, by the strong maximum principle, and the unique continuation principle [155], we infer that $v_\infty \equiv U(\ell_- - x_1)$ (if $W \in C^2$, we can apply Theorem 5.1 in [74], as we did in Theorem 10.1). However, this contradicts the second relation in (10.16).

The proof of the proposition is complete. $\square$

**Remark 10.7.** In light of the multiple-end solutions to the equation $\Delta u + u - u^3 = 0$ in the plane, constructed recently in [101] (see also [148, 172]), we infer that the uniform assumption in (10.17) is *necessary* in Theorem 10.2.

10.4. **A new proof of Gibbons' conjecture.** In this subsection, exploiting further the approach that we have developed in the previous ones, as well as the Hamiltonian structure of the equation, we will give a totally new proof of the well known Gibbons' conjecture that we mentioned in the beginning of this section.

**Theorem 10.3.** Suppose that $W \in C^2(\mathbb{R})$, $W(t) \ge 0$ for $t \in \mathbb{R}$, $W(t) > 0$ for $t \in (\mu_-, \mu_+)$, $W(\mu_\pm) = 0$, and $W''(\mu_\pm) > 0$. If $u \in C^2(\mathbb{R}^n)$ satisfies (1.22), $\mu_- \le u \le \mu_+$ in $\mathbb{R}^n$, and

$$u \to \mu_\pm, \text{ uniformly in } \mathbb{R}^{n-1}, \text{ as } x_1 \to \pm\infty, \tag{10.17}$$

then $u(x) = U(x_1)$, where $U$ satisfies

$$U'' = W'(U),\ x_1 \in \mathbb{R},\quad U(x_1) \to \mu_\pm \text{ as } x_1 \to \pm\infty. \tag{10.18}$$

*Proof.* By our assumptions on $W$, there exists a $\mu_0 \in (\mu_-, \mu_+)$ such that $W'(\mu_0) = 0$ and $W''(\mu_0) \le 0$.

By our assumptions on $u$, there exist real numbers $\ell_- \le \ell_+$ such that

$$\ell_- = \sup\{s \in \mathbb{R} :\ u(x_1, x') < \mu_0 \text{ if } x_1 < s \text{ and } x' \in \mathbb{R}^{n-1}\},$$

and

$$\ell_+ = \inf\{s \in \mathbb{R} :\ u(x_1, x') > \mu_0 \text{ if } x_1 > s \text{ and } x' \in \mathbb{R}^{n-1}\}.$$

By the strong maximum principle, we deduce that $u < \mu_0$ if $x_1 < \ell_-$ and $u > \mu_0$ if $x_1 > \ell_+$. By virtue of Theorem 10.5, in order to conclude, it suffices to show that $\ell_- = \ell_+$. We observe that, thanks to Theorem 10.1, the assertion of the current theorem follows if there exists $x' \in \mathbb{R}^{n-1}$ such that $u(\ell_-, x') = \mu_0$ or $u(\ell_+, x') = \mu_0$. So, let us assume that there exist sequences $\pm(x_\pm)_j > \pm\ell_\pm$ and $(x'_\pm)_j \in \mathbb{R}^{n-1}$ such that

$$(x_\pm)_j \to \ell_\pm,\quad u\left((x_\pm)_j, (x'_\pm)_j\right) \to \mu_0, \quad \text{and} \quad |(x'_\pm)_j| \to \infty \quad \text{as } j \to \infty.$$

Similarly to Theorem 10.2, passing to a subsequence, we find that

$$u\left(x_1, (x'_\pm)_j\right) \to U(x_1 - \ell_\pm) \ \text{ in } C^2_{loc}(\mathbb{R}), \ \text{ as } j \to \infty, \tag{10.19}$$

for some $U$ that satisfies (10.18) and $U(0) = 0$.

Since $W''(\mu_\pm) > 0$, it is standard to show that there exist constants $c, C > 0$ such that

$$|u(x) - \mu_\pm| + \left|\partial^k_{x_1}\partial^\alpha_{x'} u\right| \le Ce^{-c|x_1|}, \ \ \forall\, x = (x_1, x') \in \mathbb{R}^n, \tag{10.20}$$

for $k = 0, 1, 2, 3$ and multi-indexes $\alpha$ with $|\alpha| \le 3$ such that $k + |\alpha| > 0$. So, similarly to [66, 146] (see also [251]), we can justify integration by parts and establish the following identity of "Hamiltonian" type:

$$\nabla_{x'} g(u; x') = -h(u; x') \ \text{ for every } x' \in \mathbb{R}^{n-1},$$

where

$$g(u; x') = \int_{\mathbb{R}} x_1 \left\{\frac{1}{2}u^2_{x_1} - \frac{1}{2}|\nabla_{x'} u|^2 + W(u)\right\} dx_1 \ \text{ and } \ h(u; x') = \int_{\mathbb{R}} u_{x_1} \nabla_{x'} u dx_1.$$

Indeed, with the obvious notation, we have

$$\nabla_{x'} g = \int_{\mathbb{R}} x_1 \left\{u_{x_1} \left(\nabla_{x'} u\right)_{x_1} - \Delta_{x'} u \nabla_{x'} u + W'(u)\nabla_{x'} u\right\} dx_1.$$

We find that

$$\begin{aligned} \textstyle\int_{\mathbb{R}} x_1 u_{x_1} \left(\nabla_{x'} u\right)_{x_1} dx_1 &= \textstyle\int_{\mathbb{R}} \left\{\partial_{x_1}\left(x_1 u_{x_1} \nabla_{x'} u\right) - x_1 u_{x_1 x_1} \nabla_{x'} u - u_{x_1} \nabla_{x'} u\right\} dx_1 \\ &= \textstyle\int_{\mathbb{R}} \left\{-x_1 u_{x_1 x_1} \nabla_{x'} u - u_{x_1} \nabla_{x'} u\right\} dx_1. \end{aligned}$$

It follows that

$$\begin{aligned} \nabla_{x'} g &= \textstyle\int_{\mathbb{R}} x_1 \left\{-u_{x_1 x_1} - \Delta_{x'} u + W'(u)\right\} \nabla_{x'} u dx_1 - \int_{\mathbb{R}} u_{x_1} \nabla_{x'} u dx_1 \\ &= -\textstyle\int_{\mathbb{R}} u_{x_1} \nabla_{x'} u dx_1, \end{aligned}$$

as claimed. Next, motivated from [66], [146], we will show that $g$ is a constant (the main difference is that $x'$ was one-dimensional in the aforementioned papers). To this end, we claim that

$$\operatorname{div}_{x'} h(u; x') = 0 \ \ \forall\, x' \in \mathbb{R}^{n-1}.$$

To see this, we write $h = (h_2, \cdots, h_n)$, where

$$h_i(u; x') = \int_{\mathbb{R}} u_{x_1} u_{x_i} dx_1, \ \ i = 2, \cdots, n.$$

For $i = 2, \cdots, n$, we find that

$$\partial_{x_i} h_i = \int_{\mathbb{R}} \left(u_{x_i x_1} u_{x_i} + u_{x_1} u_{x_i x_i}\right) dx_1 = \int_{\mathbb{R}} u_{x_1} u_{x_i x_i} dx_1.$$

So, we obtain that

$$\operatorname{div}_{x'} h = \int_{\mathbb{R}} u_{x_1} \Delta_{x'} u dx_1 = \int_{\mathbb{R}} u_{x_1} \left\{W'(u) - u_{x_1 x_1}\right\} dx_1 = 0,$$

as desired. It follows that

$$\Delta_{x'} g = 0 \ \text{ in } \ \mathbb{R}^{n-1}.$$

Since $g$ is bounded (from (10.20)), by Liouville's theorem (see [142, 121]), we infer that $g$ is a constant. In particular, we have that

$$g\left(u;(x'_-)_j\right)=g\left(u;(x'_+)_j\right),\quad j\geq 1.$$

Using (10.19), (10.20), and letting $j\to\infty$, with the help of Lebesgue's dominated convergence theorem, we arrive at

$$\int_{\mathbb{R}} x_1\left\{\frac{1}{2}\left[U'(x_1-\ell_-)\right]^2+W\left(U(x_1-\ell_-)\right)\right\}dx_1=\int_{\mathbb{R}} x_1\left\{\frac{1}{2}\left[U'(x_1-\ell_+)\right]^2+W\left(U(x_1-\ell_+)\right)\right\}dx_1.$$

We conclude that $\ell_-=\ell_+$ as desired.

The proof of the theorem is complete. □

**Remark 10.8.** Related Hamiltonian identities (with $x'$ one-dimensional) can be found in [144] (see also [18, 19] for a slightly different viewpoint which employs a stress energy tensor, and [103]).

**Remark 10.9.** The original proofs in [31, 42, 118] work with the weaker conditions that $W'$ is increasing near $\mu_\pm$ and $W\in C^{1,1}$.

**Remark 10.10.** For a proof of the conjecture, when the nonlinearity $W'$ is discontinuous, we refer to [119].

**Remark 10.11.** A parabolic version of Gibbons' conjecture can be found in [44].

**Remark 10.12.** Proofs of analogs of the Gibbons' conjecture, involving more general operators than the usual Laplacian (such as fully nonlinear elliptic differential operators or fractional Laplacians), can be found in [53] and [127].

**Remark 10.13.** A Gibbons' type of conjecture for entire solutions with algebraic growth of a semilinear elliptic system, arising in Bose-Einstein condensation, was proven recently in [128].

## 11. One-dimensional symmetry in convex cylindrical domains

In [79, 117], the authors considered energy minimizing solutions to

$$\Delta u+u-u^3=0 \text{ in } \mathbb{R}\times\omega;\ \frac{\partial u}{\partial\nu}=0 \text{ on } \mathbb{R}\times\partial\omega, \tag{11.1}$$

such that

$$u\to\pm 1, \text{ uniformly for } x'\in\bar{\omega}, \text{ as } x_1\to\mp\infty,$$

where $\omega$ is a smooth bounded domain of $\mathbb{R}^{n-1}$ and $\nu$ denotes $(\mathbb{R}\times\partial\omega)$'s outer unit normal vector (in fact, they studied minimizers of the energy with $\omega$ merely bounded, without looking at the Euler-Lagrange equation). Using a rearrangement argument, they showed that $u$ is one-dimensional (see also [63] and [162]). Related results can be found in [41].

Surprisingly enough, if $n=2$, our Proposition 10.1 implies that the limit in *just one direction* is needed to reach the same conclusion *without* even assuming that $u$ is an energy minimizing solution. In this section, following the strategy of the previous section, we will show that the same property holds true in any dimension, provided that $\omega$ is convex. We emphasize that our approach applies to equations with more general nonlinearities and does not make use of the oddness of the nonlinearity in hand. It suffices that $W\in C^2$, $W>0$ in $t\in(\mu_-,\mu)$ and $W(\mu_-)=W(\mu)=0$.

11.1. **A gradient bound in convex cylindrical domains.** In order to apply the strategy of Section 10, we will first prove that the gradient bound (2.57) continues to hold in this setting. For the corresponding problem with Dirichlet boundary conditions, this was shown recently in [125]. As in the latter reference, we will follow the lines that were set in [74] for the whole space problem, with the necessary modifications in order to deal with the presence of the boundary. To this end, the authors of [125] introduced (among other things) the idea to translate the domain. Our proof is essentially the same, however we keep the domain fixed and, in contrast to the Dirichlet boundary condition case, we have to appeal to a result in [236] (originally due to [80, 188]).

**Proposition 11.1.** Let $\Omega = \Omega_0 \times \mathbb{R}^{n-n_0}$, where $\Omega_0 \subset \mathbb{R}^{n_0}$ is a bounded, smooth ($\partial\Omega$ at least $C^2$) and convex domain, and $1 \leq n_0 < n$. Let $u \in C^2(\bar{\Omega}) \cap L^\infty(\Omega)$ be a solution to

$$\Delta u - W'(u) = 0, \ x \in \Omega; \ \frac{\partial u}{\partial \nu} = 0, \ x \in \partial\Omega, \tag{11.2}$$

where $\nu$ denotes the outer unit normal vector to $\partial\Omega$, and $W \in C^2(\mathbb{R})$. If $W(t) \geq 0$, $t \in \mathbb{R}$, then

$$|\nabla u|^2 - 2W(u) \leq 0, \quad x \in \Omega.$$

*Proof.* Let $u$ be as in the assertion of the proposition. We set

$$\mathcal{F} = \left\{ v \in C^2(\bar{\Omega}) \text{ solutions of (11.2) with } |v| \leq \|u\|_{L^\infty(\Omega)} \text{ on } \bar{\Omega} \right\}.$$

Clearly $u \in \mathcal{F}$. Next, let

$$P(v,x) = |\nabla v(x)|^2 - 2W\left(v(x)\right), \quad v \in \mathcal{F}, \ x \in \bar{\Omega}.$$

These type of $P$-functions have been extensively investigated in the PDE literature (see Chapter 5 in [233]).

By formula (2.7) in [74], for $v \in \mathcal{F}$ we have

$$|\nabla v(x)|^2 \Delta P(v,x) - 2W'\left(v(x)\right) \nabla v(x) \cdot \nabla P(v,x) \geq \frac{|\nabla P(v,x)|^2}{2} \ \text{ if } x \in \Omega \text{ and } \nabla v(x) \neq 0. \tag{11.3}$$

Moreover, we find

$$\frac{\partial}{\partial \nu} P(v,x) = \frac{\partial}{\partial \nu}\left(|\nabla v|^2\right) - 2W'(v)\frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\left(|\nabla v|^2\right) \quad \text{on } \partial\Omega.$$

Since $\Omega$ is smooth and convex, and $v \in C^2(\bar{\Omega})$ satisfies $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, it follows from Lemma 2.2 in [236] (see also Lemma 2.1 in [6], [80], Theorem 1.1 in [178], Lemma 5.3 in [188], and page 79 in [233]) that

$$\frac{\partial}{\partial \nu}\left(|\nabla v|^2\right) \leq 0 \ \text{ on } \partial\Omega.$$

In turn, this implies that

$$\frac{\partial}{\partial \nu} P(v,x) \leq 0 \ \text{ on } \partial\Omega \text{ for every } v \in \mathcal{F}. \tag{11.4}$$

Now, we consider

$$P_0 \equiv \sup_{\substack{v \in \mathcal{F} \\ x \in \bar{\Omega}}} P(v,x).$$

By elliptic regularity theory (see page 24 in [196]), for $\alpha \in (0,1)$, there exists a constant $C>0$ such that

$$\|v\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \ \text{ for all } v \in \mathcal{F}. \tag{11.5}$$

Hence, it follows that $P_0$ is finite, i.e. $P_0 \in \mathbb{R}$. The proposition will be proved if we show that

$$P_0 \leq 0.$$

To this end, we will argue by contradiction, namely we assume that

$$P_0 > 0.$$

We then take $v_k \in \mathcal{F}$ and $x_k \in \bar{\Omega}$ such that

$$P_0 - \frac{1}{k} \leq P(v_k, x_k) \leq P_0, \quad k \geq 1. \tag{11.6}$$

We write

$$x_k = (y_k, z_k) \in \bar{\Omega}, \quad \text{where } y_k \in \bar{\Omega}_0, \ z_k \in \mathbb{R}^{n-n_0},$$

and set

$$u_k(x) = v_k\left(x + (0, z_k)\right), \quad x \in \bar{\Omega}.$$

Making use of (11.5), passing to a subsequence, we may assume that

$$u_k \to u_\infty \ \text{ in } C^2_{loc}(\bar{\Omega}),$$

for some $u_\infty \in C^2(\bar{\Omega})$, satisfying (11.2), with $|u_\infty| \leq \|u\|_{L^\infty(\Omega)}$ on $\bar{\Omega}$. In particular, we have that

$$u_\infty \in \mathcal{F}.$$

We may further assume that

$$y_k \to y_\infty \in \bar{\Omega}_0.$$

From (11.6), we obtain that

$$P(u_\infty, x_\infty) = P_0, \quad \text{where } x_\infty = (y_\infty, 0) \in \bar{\Omega}.$$

Consider the set

$$\mathcal{U} = \left\{ x \in \bar{\Omega} \text{ such that } P(u_\infty, x) = P_0 \right\}.$$

We already know that $\mathcal{U}$ is nonempty (because $x_\infty \in \mathcal{U}$). Moreover, since $u_\infty \in C^2(\bar{\Omega})$, it follows that

$$\mathcal{U} \text{ is relatively closed in } \bar{\Omega}. \tag{11.7}$$

We plan to prove that

$$\mathcal{U} \text{ is relatively open in } \bar{\Omega}. \tag{11.8}$$

Let $x_0 \in \mathcal{U}$. Firstly, since $W \geq 0$, observe that

$$|\nabla u_\infty(x_0)|^2 = P_0 + 2W\left(u_\infty(x_0)\right) \geq P_0 > 0.$$

So, there exists an $r>0$ such that

$$|\nabla u_\infty(x)| > 0, \quad x \in B_r(x_0) \cap \bar{\Omega}.$$

It then follows from (11.3) that

$$\Delta P(u_\infty, x) - 2\frac{W'\left(u_\infty(x)\right)}{|\nabla u_\infty(x)|^2} \nabla u_\infty(x) \cdot \nabla P(u_\infty, x) \geq 0, \quad x \in B_r(x_0) \cap \bar{\Omega}.$$

Keep in mind that

$$P(u_\infty, x) \leq P_0, \ x \in \bar{\Omega} \ \text{ and } \ P(u_\infty, x_0) = P_0.$$

Two cases can occur:

- If $x_0 \in \Omega$, it follows at once from the strong maximum principle that
$$P(u_\infty, x) = P_0, \quad x \in B_r(x_0) \cap \bar{\Omega},$$
namely $B_r(x_0) \cap \bar{\Omega} \subset \mathcal{U}$;
- If $x_0 \in \partial\Omega$, by (11.4) and Hopf's boundary point lemma, we are led again to the same conclusion.

Thus, we have shown that relation (11.8) holds.

By (11.7), (11.8), and the connectedness of $\bar{\Omega}$, we conclude that
$$\mathcal{U} = \bar{\Omega}.$$
In other words, we have arrived at
$$|\nabla u_\infty(x)|^2 = P_0 + 2W\left(u_\infty(x)\right) \geq P_0 > 0, \quad x \in \bar{\Omega}. \tag{11.9}$$
We will show that this comes in contradiction with the fact that $u_\infty$ is bounded. We fix a $Q \in \Omega$ and consider the gradient flow
$$\begin{cases} \gamma'(t) = \nabla u_\infty\left(\gamma(t)\right), \\ \gamma(0) = Q. \end{cases}$$
We note that $\gamma$ is globally defined since $\nabla u_\infty \in L^\infty(\Omega)$ and $\gamma$ cannot hit $\partial\Omega$ due to $\frac{\partial u_\infty}{\partial \nu} = 0$ on $\partial\Omega$. We have
$$\frac{d}{dt}\left[u_\infty\left(\gamma(t)\right)\right] = \nabla u_\infty\left(\gamma(t)\right) \cdot \gamma'(t) = \left|\nabla u_\infty\left(\gamma(t)\right)\right|^2 \overset{(11.9)}{\geq} P_0 > 0$$
Thus, we get
$$u_\infty\left(\gamma(t)\right) \geq u_\infty(Q) + P_0 t, \quad t \geq 0,$$
which implies that $u_\infty$ is unbounded; a contradiction.

The proof of the proposition is complete. □

11.2. **The symmetry result.** Our main result in this section is the following:

**Proposition 11.2.** Let $u$ be a nonconstant bounded solution to (11.1) such that $u \to 1$ as $x_1 \to -\infty$ uniformly for $x' \in \bar{\omega}$. If $\omega$ is smooth and convex, then $u$ is one dimensional.

*Proof.* Given $R > 0$, let $u_R$ be as in Lemma 2.1, with $n = 1$ and $W'(t) = t^3 - t$ (i.e. $\mu = 1$). By the uniform in $\bar{\omega}$ asymptotic behavior of $u$ as $x_1 \to -\infty$, we infer that there exists a large $M > R$ such that $u(x) > u_R(0)$ if $x \in \bar{\Omega} \cap \{x_1 \leq -M + R\}$, i.e.
$$u > u_R(x_1 + M) \ \text{ on } \bar{\Omega} \cap \left\{|x_1 + M| \leq R\right\}. \tag{11.10}$$
Let
$$\underline{u}_{R,Q_1}(x_1, x') = \begin{cases} u_R(x_1 + Q_1), & |x_1 + Q_1| < R, \ x' \in \omega, \\ 0, & \text{otherwise}. \end{cases} \tag{11.11}$$
Note that
$$\frac{\partial \underline{u}_{R,Q_1}}{\partial \nu} = 0 \ \text{ on } \partial\Omega \text{ if } x_1 + Q_1 \neq \pm R.$$
We claim that $u$ vanishes at some point on $\mathbb{R} \times \bar{\omega}$. Indeed, if not, we can move $Q_1$ in $\mathbb{R}$ to find that $u > u_R(0)$ in $\bar{\Omega}$ for all $R > 0$, by the sliding method (note also that $\underline{u}_{R,Q_1}$ and

$u$ cannot touch on $x_1 + Q_1 = \pm R$). In view of the obvious analog of (2.3), this implies that $u \equiv 1$ which cannot happen since $u$ is assumed to be nonconstant.

Now, by virtue of the uniform asymptotic behavior of $u$ as $x_1 \to -\infty$, we may assume without loss of generality that

$$u > 0 \text{ if } x_1 < 0 \text{ and } u(P) = 0 \text{ at some } P = (0, P') \text{ with } P' \in \bar{\omega}, \tag{11.12}$$

(because (11.1) is invariant with respect to translations in the $x_1$-direction). So, in view of (11.10), we can slide $\underline{u}_{R,Q_1}$ along the $x_1$-axis (decreasing $Q_1$), staying below the graph of $u$, until we reach

$$u(x) > \underline{u}_{R,R}(x), \quad x \in \{x_1 < 0\} \times \bar{\omega}.$$

In particular, recalling (11.12), we obtain that

$$u(x_1, x') > u_R(x_1 + R), \quad -R < x_1 < 0, \; x' \in \bar{\omega}, \text{ and } u(0, P') = 0 = u_R(R), \text{ where } P' \in \bar{\omega}. \tag{11.13}$$

As in Proposition 10.6 (see in particular (10.12)), via the obvious analog of (2.3) (loosely speaking, letting $R \to \infty$ in (11.13)), we obtain that

$$u(x) \geq \mathbf{U}(-x_1) \text{ on } \{x_1 \leq 0\} \times \bar{\omega}, \text{ and } u = \mathbf{U} = 0 \text{ at } P = (0, P'), \tag{11.14}$$

where $\mathbf{U}$ as in (1.12).

Two cases can occur:

- If $P' \in \omega$, by Hopf's boundary point lemma (applied to the equation for $u - \mathbf{U}(-x_1)$), we deduce that either

$$u_{x_1} < -\mathbf{U}'(0) = -\sqrt{2W(0)} \text{ at } P = (0, P') \in \Omega,$$

  or $u \equiv \mathbf{U}(-x_1)$ on $\{x_1 \leq 0\} \times \bar{\omega}$. Since $\omega$ is convex and $u$ is bounded, as in Proposition 10.6, the former scenario cannot happen by virtue of the gradient bound in Proposition 11.1 (this is the first time in the proof that we used the convexity of $\omega$). We therefore must have that $u(x) = \mathbf{U}(-x_1)$ on $\{x_1 \leq 0\} \times \omega$ and, by the unique continuation principle [155] (applied to the equation for $u - \mathbf{U}(-x_1)$), we conclude that $u \equiv \mathbf{U}(-x_1)$ in $\Omega$, as desired.
- If $P' \in \partial\omega$, from (11.14) and the gradient bound of Proposition 11.1, we obtain that

$$u_{x_1}(0, P') = -\sqrt{2W(0)}. \tag{11.15}$$

  Actually, by the strong maximum principle and Hopf's boundary point lemma, unless $u \equiv \mathbf{U}(-x_1)$, there is strict inequality in (11.14) at points in $\{x_1 < 0\} \times \bar{\omega}$. In the latter case, we would like to employ Hopf's boundary point lemma to get $u_{x_1}(0, P') < -\sqrt{2W(0)}$, which contradicts (11.15). However, this time we cannot fit a ball in $\{x_1 < 0\} \times \omega$ which is tangent to $P$. Nevertheless, with a little care, we can adapt the standard proof of Hopf's boundary point lemma to cover the situation at hand, where the point is on a corner of the boundary of the domain $\{x_1 < 0\} \times \omega$. Indeed, let

$$\varphi = \mathbf{U}(-x_1) - u. \tag{11.16}$$

  We have

$$\Delta\varphi - c(x)\varphi = 0, \; \varphi < 0 \text{ in } \{x_1 < 0\} \times \omega,$$

  for some bounded function $c$, say $|c(x)| < d$, and $\varphi(0, P') = 0$. For $a > 0$ to be determined, let

$$v(x) = v(x_1, x') = e^{-a(x_1+1)} - e^{-a} > 0, \quad x_1 \in (-1, 0), \; x' \in \omega.$$

We can choose $a > 0$ sufficiently large ($a > \sqrt{d}$) so that

$$\Delta v - dv > 0 \text{ on } [-1,0] \times \bar{\omega}. \tag{11.17}$$

Now, let

$$\tilde{v} = \frac{\ell}{2v(-1)} v < 0, \quad \text{where } \ell = \max_{x_1=-1} \varphi < 0.$$

It follows that $\tilde{v}$ satisfies

$$\Delta\tilde{v} - d\tilde{v} < 0 \text{ on } [-1,0] \times \bar{\omega}, \ \frac{\partial\tilde{v}}{\partial\nu} = 0 \text{ on } [-1,0] \times \partial\omega,$$

$$\tilde{v} = \frac{\ell}{2} < 0 \text{ on } x_1 = -1; \ \tilde{v} = 0 \text{ and } \tilde{v}_{x_1} > 0 \text{ on } x_1 = 0. \tag{11.18}$$

We claim that

$$\varphi - \tilde{v} \leq 0 \text{ on } [-1,0] \times \bar{\omega}. \tag{11.19}$$

We have that

$$\varphi - \tilde{v} \leq 0 \text{ on } x_1 = -1 \text{ and } x_1 = 0, \text{ and } \frac{\partial(\varphi - \tilde{v})}{\partial\nu} = 0 \text{ on } [-1,0] \times \partial\omega.$$

Suppose that (11.19) does not hold, namely that the maximum of $\varphi - \tilde{v}$ over $[-1,0] \times \bar{\omega}$ is positive and is achieved at some $x_0 \in (-1,0) \times \bar{\omega}$. Note that

$$\Delta(\varphi - \tilde{v}) > c(x)\varphi - d\tilde{v} > (c(x) - d)\,\varphi > 0 \text{ at } x = x_0,$$

(we have silently used that $\varphi, \tilde{v} \in C^2(\bar{\Omega})$ in case $x_0 \in \partial\Omega$). By the maximum principle, we deduce that $x_0$ cannot be in $(-1,0) \times \omega$. Moreover, by the usual Hopf's boundary point lemma, the point $x_0$ can neither be in $(-1,0) \times \partial\omega$. We have thus been led to a contradiction, which means that relation (11.19) holds true. It follows in particular that the restriction of $\varphi - \tilde{v}$ on the line $\{-1 \leq x_1 \leq 0\} \times \{P'\}$ attains its maximum value at $x_1 = 0$, which implies, via (11.16), that

$$u_{x_1}(0,P') \leq -\mathbf{U}'(0) - \tilde{v}_{x_1}(0,P') \overset{(11.18)}{<} -\mathbf{U}'(0) = -\sqrt{2W(0)}.$$

Recalling that $u(0,P') = 0$, the above relation contradicts (11.15). We conclude again that $u \equiv \mathbf{U}(-x_1)$, as desired.

The proof of the proposition is complete. ☐

**Remark 11.1.** In analogy to Theorem 10.2, we have the following:
Let $u$ be a bounded solution to (11.1) such that $u \to 1$ as $x_1 \to \pm\infty$ uniformly for $x' \in \bar{\omega}$. If $\omega$ is smooth and convex, then $u \equiv 1$.

## Appendix A. Some useful "comparison" lemmas of the calculus of variations

The following is essentially Lemma 2.1 in [134].

**Lemma A.1.** Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and let $v \in W^{1,2}(\mathcal{O})$. Define $\tilde{v} : \mathcal{O} \to \mathbb{R}$ as

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } v(x) \in [0,\mu], \\ \mu & \text{if } v(x) \in (-\infty,-\mu) \cup (\mu,\infty), \\ -v(x) & \text{if } v(x) \in (-\mu,0). \end{cases}$$

Then $\tilde{v} \in W^{1,2}(\mathcal{O})$ and, if $W$ is $C^2$ and satisfies **(a')**, we have

$$\int_{\mathcal{O}} \left\{ \frac{1}{2}|\nabla \tilde{v}|^2 + W(\tilde{v}) \right\} dx \leq \int_{\mathcal{O}} \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx.$$

*Proof.* (Sketch) Firstly, note that $\tilde{v} = G(v)$, $x \in \mathcal{O}$, for some Lipschitz (piecewise linear) function $G : \mathbb{R} \to \mathbb{R}$. Thus, $\tilde{v} \in W^{1,2}(\mathcal{O})$, see for instance [115]. Then, to finish, note that

$$|\nabla \tilde{v}| \leq |\nabla v| \ \text{ and, thanks to } \mathbf{(a')}, \ \ W(\tilde{v}) \leq W(v) \ \text{ a.e. in } \mathcal{O}, \tag{A.1}$$

(the former inequality may be proven as in page 93 in [165]). □

The following is an extension of Lemma A.1, and is motivated from [14, 20] (see also [15] for an extension). Our proof follows [234].

**Lemma A.2.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with Lipschitz boundary, and $W : \mathbb{R} \to \mathbb{R}$ be a $C^2$ potential such that conditions **(a')** and (2.25) hold. Further, let $\mathcal{A} \subset \Omega$ be a bounded domain with Lipschitz boundary such that $\bar{\mathcal{A}} \subset \Omega$. Moreover, assume that

- $u \in W^{1,2}(\Omega)$, $0 \leq u \leq \mu$ a.e. in $\Omega$
- $\mu - u \leq \eta$ a.e. on $\partial\mathcal{A}$, in the sense of Sobolev traces (see [115]), for some $\eta \in \left(0, \frac{d}{2}\right)$.

Then, there exists $\tilde{u} \in W^{1,2}(\Omega)$ such that

$$\begin{cases} \tilde{u}(x) = u(x), \quad x \in \Omega \backslash \mathcal{A}, \\ \\ \mu - \eta \leq \tilde{u}(x) \leq \mu, \quad x \in \mathcal{A}, \\ \\ \int_{\Omega} \left\{ \frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u}) \right\} dx \leq \int_{\Omega} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx. \end{cases} \tag{A.2}$$

If condition (2.25) holds with strict inequality, and there exists a set $\mathcal{B} \subset \mathcal{A}$ of nonzero measure such that

$$u < \mu - \eta \ \text{ a.e. on } \ \mathcal{B},$$

then the last relation in (A.2) holds with a strict inequality.

*Proof.* (Sketch) The first assertion of the lemma can be deduced similarly to Lemma A.1. Indeed, the desired function is

$$\tilde{u}(x) = \begin{cases} \min\left\{\mu, \max\left\{u(x),\ 2\mu - 2\eta - u(x)\right\}\right\}, & x \in \mathcal{A}, \\ \\ u(x), & x \in \Omega \backslash \mathcal{A}. \end{cases} \tag{A.3}$$

We point out that $\tilde{u} \in W^{1,2}(\mathcal{A})$ similarly to Lemma A.1, and $\tilde{u} \in W_0^{1,2}(\Omega)$ because $\mathcal{A}$ has Lipschitz boundary and $\tilde{u} = u$ on $\partial\mathcal{A}$ in the sense of Sobolev traces (see again [115]). Note that if $\mu - 2\eta \leq u(x) \leq \mu$ then $\mu - d < u(x) \leq \tilde{u}(x) \leq \mu$, so relation (2.25) implies that $W(\tilde{u}(x)) \leq W(u(x))$. Furthermore, if $0 \leq u(x) \leq \mu - 2\eta$ then $\tilde{u}(x) = \mu$ and $W(\tilde{u}(x)) = 0 \leq W(u(x))$. Also keep in mind the first relation in (A.1).

The second assertion can be shown with a little more care. Replacing $u$ by the minimizer of the corresponding energy functional $J(\cdot;\mathcal{A})$ (recall (2.1)) among functions $v \in W^{1,2}(\mathcal{A})$ such that $v - u \in W_0^{1,2}(\mathcal{A})$, we may assume that $u$ is a smooth solution of (3.1) in $\mathcal{A}$. Firstly, we consider the case where

$$\mu - 2\eta \leq u(x) \leq \mu \ \text{ on } \bar{\mathcal{A}}.$$

In that case, we have that $\mu - d < u < \tilde{u} \leq \mu$ on $\mathcal{B}$. In turn, from the assumption that the inequality in (2.25) is strict, we obtain that $W(\tilde{u}) < W(u)$ on $\mathcal{B}$. Since the set $\mathcal{B}$ has positive measure, taking into account our previous discussion for the first assertion, we arrive at

$$\int_\Omega W(\tilde{u})dx < \int_\Omega W(u)dx. \tag{A.4}$$

Hence, the second assertion holds in this case. On the other side, if

$$0 \leq u(x_0) < \mu - 2\eta \quad \text{for} \quad \text{some } x_0 \in \mathcal{A},$$

then $0 \leq u \leq \mu - 2\eta \leq \tilde{u} = \mu$ in an open neighborhood of $x_0$. In this neighborhood, via **(a')**, we have that $W(u) \geq \min_{t\in[0,\mu-2\eta]} W(t) > 0$ while $W(\tilde{u}) = 0$. It follows that relation (A.4) holds in this case as well. Keeping in mind the first relation in (A.1), we conclude that the second assertion of the lemma holds.

The sketch of proof of the lemma is complete. □

The following is Lemma 2.3 in [96], which is reproduced in Lemma 1 in [173] and Lemma 2.1 in [179], see also Theorem 1.4 in [133] and Lemma 3.1 in [159].

**Lemma A.3.** Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let $g_1(x,t), g_2(x,t)$ be locally Lipschitz functions with respect to $t$, measurable functions with respect to $x$, and for any bounded interval $I$ there exists a constant $C$ such that $\sup_{x\in\mathcal{D},t\in I}|g_i(x,t)| \leq C$, $i = 1, 2$, holds. Let

$$G_i(x,t) = \int_0^t g_i(x,s)ds, \ i = 1, 2.$$

For $\eta_i \in W^{1,2}(\mathcal{D}),\ i = 1, 2$, consider the minimization problem:

$$\inf\left\{J_i(u;\mathcal{D}) \mid u - \eta_i \in W_0^{1,2}(\mathcal{D})\right\}, \quad \text{where} \quad J_i(u;\mathcal{D}) = \int_{\mathcal{D}} \left\{\frac{1}{2}|\nabla u|^2 - G_i(x,u)\right\} dx.$$

Let $u_i \in W^{1,2}(\mathcal{D}),\ i = 1, 2$, be minimizers to the minimization problems above. Assume that there exist constants $m < M$ such that

- $m \leq u_i(x) \leq M$ a.e. for $i = 1, 2,\ x \in \mathcal{D}$,
- $g_1(x,t) \geq g_2(x,t)$ a.e. for $x \in \mathcal{D},\ t \in [m, M]$,
- $M \geq \eta_1(x) \geq \eta_2(x) \geq m$ a.e. for $x \in \mathcal{D}$.

Suppose further that $\eta_i \in W^{2,p}(\mathcal{D})$ for any $p > 1$, and that they are *not identically equal* on $\partial\mathcal{D}$. Then, we have

$$u_1(x) \geq u_2(x), \quad x \in \mathcal{D}.$$

## Appendix B. A Liouville-type theorem

Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and

$$\begin{cases} f(0) = 0, \\ f(t) > 0,\ t > 0, \\ f \text{ is non–decreasing and convex on } [0,\infty), \\ \int_{t_0}^{\infty}\left[\int_{t_0}^{t} f(s)ds\right]^{-\frac{1}{2}} dt < \infty \quad \forall\ t_0 > 0. \end{cases}$$

In the mathematical literature, the above integral condition is known as Keller-Osserman condition, see [121], [163] and [200]. These conditions are clearly satisfied for

$$f(t) = t|t|^{p-1} \text{ with } p > 1. \tag{B.1}$$

The following is Theorem 4.7 in the review article [121]. As we have already discussed at the end of Remark 4.1, it was originally proven in [58] for the special case of the power nonlinearity (B.1).

**Theorem B.1.** Let $f$ satisfy the above properties.

**(i):** Suppose $u \in L^1_{loc}(\mathbb{R}^n)$ is such that $f(u) \in L^1_{loc}(\mathbb{R}^n)$ and

$$-\Delta u + f(u) \leq 0 \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ (distributionally)}.$$

Then $u \leq 0$ a.e. on $\mathbb{R}^n$.

**(ii):** Assume also that $f$ is an odd function. Suppose $u \in L^1_{loc}(\mathbb{R}^n)$ is such that $f(u) \in L^1_{loc}(\mathbb{R}^n)$ and

$$-\Delta u + f(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

Then $u = 0$ a.e. on $\mathbb{R}^n$.

## Appendix C. A doubling lemma

The following is a very useful result from [206].

**Lemma C.1.** Let $(X, d)$ be a complete metric space, $\Gamma \subset X$, $\Gamma \neq X$, and $\gamma : X\backslash\Gamma \to (0, \infty)$. Assume that $\gamma$ is bounded on all compact subsets of $X\backslash\Gamma$. Given $k > 0$, let $y \in X\backslash\Gamma$ be such that

$$\gamma(y)\text{dist}(y, \Gamma) > 2k.$$

Then, there exists $x \in X\backslash\Gamma$ such that

- $\gamma(x)\text{dist}(x, \Gamma) > 2k$,
- $\gamma(x) \geq \gamma(y)$,
- $2\gamma(x) \geq \gamma(z) \quad \forall\, z \in B_{\frac{k}{\gamma(x)}}$.

We remark that this doubling lemma is proven similarly as Baire's category theorem.

## Appendix D. Some remarks on equivariant entire solutions to a class of elliptic systems of the form $\Delta u = W_u(u)$, $u : \mathbb{R}^n \to \mathbb{R}^n$

In this appendix, motivated from Remark 2.9, we indicate how to simplify some arguments of the recent paper [13], in the case of the equations that are considered there as representative examples.

We will use exactly the same notation of [13]. This appendix should be read with a copy of [13] at hand.

In [13], the author provides a simpler proof of the recent result in [12], concerning the existence of equivariant entire solutions to a class of semilinear elliptic systems of the form $\Delta u = W_u(u)$, $u : \mathbb{R}^n \to \mathbb{R}^n$ (the same approach applies for the case $u : \mathbb{R}^n \to \mathbb{R}^m$). Besides of assuming that $W_u$ is also equivariant, the main assumption in the latter papers, which was subsequently removed completely in [135] (recall also our Theorem 1.2 for more general results in the scalar case), is that of "$Q$-monotonicity" (this essentially corresponds to assumption **(b)** from our introduction). In all the examples of $W$'s found in [12, 13], for which this assumption could be verified, the function $Q$ was plainly

$$Q(u) = |u - a_1|,$$

(this was also the case for the example in [16]). In the sequel, except from Remark D.1, we will assume this choice of $Q$.

We may assume that

$$W(u) \geq c^2|u - a_1|^2 = c^2Q^2(u), \quad u \in D \cap B_M, \tag{D.1}$$

because $a_1 \in \mathbb{R}^n$ is the only minimum of $W \in C^2$ in $D$, $W > 0$ in $D\backslash\{a_1\}$, and $a_1$ is non-degenerate. Let $x_R \in D \cap B_{4R}$ be any point as in the beginning of Section 6 in [13] (namely with $B_{3R}(x_R) \subset D$). From Lemma 4.1 in [13], via the above relation (we have $|u_R| \leq M$), we obtain that

$$\Delta Q(u_R) \geq 0 \text{ in } D \text{ (weakly) and } \int_{B_{2R}(x_R)} Q^2\left(u_R(x)\right) dx \leq CR^{n-1},$$

for some constant $C > 0$ that is independent of $R$. Now, as in Remark 2.9 herein, we infer that

$$\sup_{B_R(x_R)} Q(u_R) \leq CR^{-n} \int_{B_{2R}(x_R)} Q\left(u_R\right) dx \leq CR^{-n}R^{\frac{n}{2}}R^{\frac{n-1}{2}} = CR^{-\frac{1}{2}} \to 0 \quad \text{as } R \to \infty,$$

($C$ is again independent of $R$). Let us mention that Section 6 in [13] was devoted to proving a similar relation (in fact, weaker but without making use of (D.1)) using De Giorgi's oscillation lemma and an iteration scheme.

**Remark D.1.** It seems likely that the use of the function $\vartheta$ in [13] can be avoided (as well as the covering argument of [134]) in order to extend the domain of validity of the above bound. To support this, we note that the function $Q(u_R)$ is a weak lower solution to a problem of the form

$$\Delta u = f(u) = \begin{cases} c^2(a-u), & 0 \leq u \leq a, \\ \\ 0, & u \geq a, \end{cases} \tag{D.2}$$

in $D$ (see also the relation between (45) and (46) in [11]). As we observed in our introduction, the continuous patching of the radial comparison functions, analogously to (1.7) (after reflecting them), together with zero can form a weak upper solution to (D.2) which may also be chosen to lie above $Q(u_R)$ on $B_R(x_R)$. Then, one can extend the domain of validity of the above estimate by plainly sliding around $x_R$ in $D$ (a fixed distance away from the boundary), using a weak version of the sliding method (the point is that the function $f$ is Lipschitz).

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